Homework # 7

§3.6 #4, 11

4/ Solve the heat problem for $L=1$, $c=1$, $f(x) = \begin{cases} 100 & 0<x\le\frac{1}{2} \\ 0 & \frac{1}{2}<x<1 \end{cases}$

PDE $\quad \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, $0<x<1$, $t>0$

BC $\quad \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(L,t) = 0$, $t>0$

IC $\quad u(x,0) = f(x) = \begin{cases} 100 & 0<x\le\frac{1}{2} \\ 0 & \frac{1}{2}<x<1 \end{cases}$.

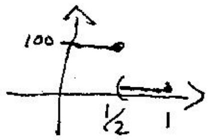
We know the solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos\frac{n\pi x}{L} \quad , \lambda_n = \frac{cn\pi}{L}$$

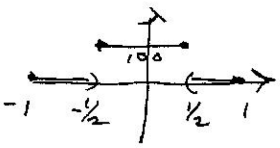
$$= a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos n\pi x \qquad \text{on } (0,1),\ t>0$$

where a_0, a_n are the coefficients of the cosine series expansion for f.

f(x) :



The even extension of f is



§2.3, exercise 6, gives the Fourier series for the even extension of f as

$$\frac{100\cdot\frac{1}{2}}{1} + \frac{200}{\pi}\sum_{n=0}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} \cos n\pi x$$

Then $\underline{a_0 = 50}$, $a_n = \dfrac{200}{n\pi}\sin\dfrac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ \pm\dfrac{200}{n\pi} & n \text{ odd} \end{cases}$

$$\underline{a_{2k+1} = \pm\frac{200}{(2k+1)\pi} = (-1)^k \frac{200}{(2k+1)\pi}}$$

$$\underline{a_{2k} = 0}$$

Example 2

// PDE $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

BC $u(0,t)=0$, $\dfrac{\partial u}{\partial x}(L,t) = -\kappa\, u(L,t)$, $t>0$

IC $u(x,0) = x$.

From the text we have the solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-c^2 \mu_n^2 t} \sin \mu_n x \quad , \quad \tan \mu_n L = -\frac{\mu_n}{\kappa}.$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \mu_n x$$

$$c_n = \frac{\int_0^L x \sin \mu_n x \, dx}{\int_0^L \sin^2 \mu_n x \, dx} = \frac{\left(\frac{1}{\mu_n^2} \sin \mu_n x - \frac{x}{\mu_n} \cos \mu_n x\right)\Big]_0^L}{\left(\frac{x}{2} - \frac{1}{4\mu_n} \sin 2\mu_n x\right)_0^L}$$

$$= \frac{\frac{1}{\mu_n^2} \sin \mu_n L - \frac{L}{\mu_n} \cos \mu_n L - \frac{1}{\mu_n^2} \cancel{\sin 0}^{\,0} + \frac{\cancel{0 \cos 0}^{\,0}}{\mu_n}}{\frac{L}{2} - \frac{1}{4\mu_n} \sin 2\mu_n L - 0 + \cancel{\frac{1}{4\mu_n} \sin 0}_{\,0}}$$

$$= \frac{\frac{1}{\mu_n^2}\left(\mu_n \sin \mu_n L - L \cos \mu_n L\right)}{\frac{L}{2} - \frac{1}{4\mu_n} \sin 2\mu_n L}$$

$$= \frac{\frac{\cos \mu_n L}{\mu_n^2}\left(\mu_n \tan \mu_n L - L\right)}{\frac{L}{2} - \frac{1}{4\mu_n} \sin 2\mu_n L}$$

$$= \frac{\frac{\cos \mu_n L}{\mu_n^2}\left(\mu_n \left(\frac{-\mu_n}{\kappa}\right) - L\right)}{\frac{2\mu_n L - \sin 2\mu_n L}{4\mu_n}} = \boxed{\frac{\frac{-4}{\mu_n} \cos \mu_n L \left(\frac{\mu_n^2}{\kappa} - L\right)}{2\mu_n L - \sin 2\mu_n L}}$$