Homework #5

Hwk 2.7 #4, 8, 10

4. $9y'' + 6y' + 10y = \sin t$

Find y_h: $9\lambda^2 + 6\lambda + 10 = 0$

$$\lambda = \frac{-6 \pm \sqrt{36-360}}{18} = -\frac{1}{3} \pm \frac{6\sqrt{1-10}}{18} = -\frac{1}{3} \pm i\,\frac{6\cdot 3}{18}$$

$$y_h = c_1 e^{-\frac{1}{3}t}\cos t + c_2 e^{-\frac{1}{3}t}\sin t \qquad = -\frac{1}{3} \pm i$$

Find y_p: $F(t) = \sin t$ $\quad a_0 = a_n = 0 \quad b_n = 0$ except for $n=1$, where $b_1 = 1$

Then $\alpha_0 = \alpha_n = 0\ \forall n$, and $\beta_n = 0$ except for $n=1$. At $n=1$, we have

$$\beta_1 = \frac{A_1 b_1 + B_1 \cancel{a_1}}{A_1^2 + B_1^2} = \frac{A_1}{A_1^2+B_1^2} = \quad \begin{pmatrix} m=9 \\ c=6 \\ k=10 \\ p=\pi \end{pmatrix}$$

$A_1 = 10 - 9(1)^2 = 1, \quad B_1 = 6(1) = 6$

$$\beta_1 = \frac{1}{1+36} = \frac{1}{37}$$

$$y_p = \frac{1}{37}\sin t$$

General solution: $\underline{y = y_h + y_p = c_1 e^{-\frac{1}{3}t}\cos t + c_2 e^{-\frac{1}{3}t}\sin t + \frac{1}{37}\sin t}$

Steady-state solution $\underline{y_s = \frac{1}{37}\sin t}$ $\quad = \lim_{t\to\infty} y = y_p$

8. $y'' + 2y' + 2y = \sin t + 2\cos 2t$ $\qquad m=1,\ c=2,\ k=2$

$F(t) = \sin t + 2\cos 2t$

Then $p = \pi$, $\quad a_0 = 0$, $a_n = 0$ except $n=2$, $b_n = 0$ except $n=1$

$a_2 = 2, \quad b_1 = 1$

Then $y_p(t) = \alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nt + \beta_n \sin nt)$

Since the α_n's depend on a_n, b_n, and β_n's depend on a_n, b_n, we

Know $\alpha_n = \beta_n = 0$, $n > 2$ and $\alpha_0 = 0$.

$$\alpha_1 = \frac{A_1 \cancel{a_1} - B_1 b_1}{A_1^2 + B_1^2} = \frac{-B_1}{A_1^2 + B_1^2} = \frac{-2}{1+4} = -\frac{2}{5}$$

$$\alpha_2 = \frac{A_2 a_2 - B_2 \cancel{b_2}}{A_2^2 + B_2^2} = \frac{2A_2}{A_2^2 + B_2^2} = \frac{-4}{2+16} = -\frac{4}{18} = \frac{-2}{9}$$

$$\beta_1 = \frac{A_1 b_1 + B_1 \cancel{a_1}}{A_1^2 + B_1^2} = \frac{A_1}{A_1^2 + B_1^2} = \frac{1}{5}$$

$$\beta_2 = \frac{\cancel{A_2 b_2} + B_2 a_2}{A_2^2 + B_2^2} = \frac{2B_2}{A_2^2 + B_2^2} = \frac{8}{18} = \frac{4}{9}$$

$A_n = k - \mu(n)^2$

$A_1 = 2 - 1 = 1$

$A_2 = 2 - 1(4) = -2$

$B_n = cn$

$B_1 = 2$

$B_2 = 4$

$$\underline{y_s = y_p = -\frac{2}{5}\cos t - \frac{2}{9}\cos 2t + \frac{1}{5}\sin t + \frac{4}{9}\sin 2t}$$

10. $$8y'' + .01y' + 15.01y = F(t) = \begin{cases} -\frac{5\pi}{2}\left(t - \frac{\pi}{2}\right) & 0 \le t \le \pi \\ \frac{5\pi}{2}\left(t - \frac{3\pi}{2}\right) & \pi \le t \le 2\pi \end{cases}$$

Natural frequency of spring $= \omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{15.01}{8}} \approx \sqrt{2} \approx 1.4$

Frequency of n^{th} normal mode $= \frac{n\pi}{P} = \frac{n\pi}{\pi} = n$

Either 1^{st} or 2^{nd} mode will dominate since $1 < \omega_0 < 2$;
probably 1^{st} mode will be dominant.