Homework #3

§2.4 #3a, 14

<u>3a</u> $f(x) = x^2$ $(0,1)$. Find the half-range expansions.

Cosine series expansion: $a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p}$ $0<x<p$

$$a_0 = \frac{1}{p}\int_0^p f, \quad a_n = \frac{2}{p}\int_0^p f \cos\frac{n\pi x}{p}$$

$$a_0 = \int_0^1 x^2 = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$$

$$a_n = 2\int_0^1 x^2 \cos n\pi x = 2x^2\frac{\sin n\pi x}{n\pi}\Big]_0^1 - 2\int_0^1 \frac{2}{n\pi}\, x \sin n\pi x\, dx$$

$u = x^2$ $dv = \cos n\pi x\, dx$ $u = x$ $dv = \sin n\pi x\, dx$

$du = 2x\,dx$ $v = \frac{\sin n\pi x}{n\pi}$ $du = dx$ $v = -\frac{\cos n\pi x}{n\pi}$

$$= 2\frac{\overset{0}{\cancel{\sin n\pi}}}{n\pi} - 0 - \frac{4}{n\pi}\left[-x\frac{\cos n\pi x}{n\pi}\Big]_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\right]$$

$$= -\frac{4}{n\pi}\left(-\frac{\cos n\pi}{n\pi} - 0\right) - \frac{4}{(n\pi)^2}\frac{\sin n\pi x}{n\pi}\Big]_0^1 = \frac{4}{(n\pi)^2} - \frac{4}{(n\pi)^3}\left(\overset{0}{\cancel{\sin n\pi}} - 0\right)$$

$$= \frac{4}{n^2\pi^2}\cos n\pi = \frac{4}{n^2\pi^2}(-1)^n$$

$$\boxed{\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x}$$

Sine series expansion: $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p}$ $(0,p)$, $b_n = \frac{2}{p}\int_0^p f \sin\frac{n\pi x}{p}$

$$b_n = 2\int_0^1 x^2 \sin n\pi x = 2x^2\left(-\frac{\cos n\pi x}{n\pi}\right)\Big|_0^1 + \frac{4}{n\pi}\int_0^1 x\cos n\pi x$$

$u = x^2$ $dv = \sin n\pi x$ $u = x$ $dv = \cos n\pi x$

$du = 2x\,dx$ $v = -\frac{\cos n\pi x}{n\pi}$ $du = dx$ $v = \frac{\sin n\pi x}{n\pi}$

$$= \frac{-2\cos n\pi}{n\pi} - 0 + \frac{4}{n\pi}\left(\frac{x \sin n\pi x}{n\pi}\Big]_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}\right)$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n\pi}\left(0 - 0 + \frac{\cos n\pi x}{(n\pi)^2}\right)\Big|_0^1$$

$$= 2\frac{(-1)^{n+1}}{n\pi} + \frac{4}{(n\pi)^3}(\cos n\pi - 1)$$

$$\boxed{\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}2}{n\pi} + \frac{4}{n^3\pi^3}\left((-1)^n - 1\right)\right]\sin n\pi x}$$

14. $f(x) = (1+\cos\pi x)\sin\pi x$ on $(0,1)$ Find sine series expansion

$$b_n = \frac{2}{p}\int_0^p f \sin\frac{n\pi x}{p}$$

$$b_n = 2\int_0^1 (1+\cos\pi x)\sin\pi x \sin n\pi x$$

$$= 2\int_0^1 \sin\pi x \sin n\pi x + 2\int_0^1 \cos\pi x \sin\pi x \sin n\pi x$$

$$= \begin{cases} 2\int_0^1 \sin^2\pi x & n=1 \\ 0 & n>1 \end{cases} + \int_0^1 \sin 2\pi x \sin n\pi x$$

$$= \begin{cases} 2\left(\frac{x}{2} - \frac{1}{4\pi}\sin 2\pi x\right)\Big|_0^1 \\ 0 & n\neq 1 \end{cases} + \begin{cases} \int_0^1 \sin^2 2\pi x & n=2 \\ 0 & n\neq 2 \end{cases}$$

Then $b_1 = 2\left(\frac{1}{2} - 0\right) = 1$

$$b_2 = \int_0^1 \sin^2 2\pi x = \frac{x}{2} - \frac{1}{8\pi}\sin 4\pi x \Big]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$b_n = 0 \quad n > 2.$

$$\sin\pi x + \frac{1}{2}\sin 2\pi x$$