Homework #10

§5.2 #2 §5.5 # 6, 7, 12, 13, 18

5.2 #2 PDE: $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta + \frac{\partial u}{\partial \theta} + \csc^2\theta \frac{\partial^2 u}{\partial \phi^2}\right) = 0$

$0 < r < 1, \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi$

BC: $u(1, \theta) = f(\theta) = \cancel{2\theta(\cos\theta + 1)}\ \cos^2\theta + 2$

Solution is $u(r, \theta) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{1}\right)^n P_n(\cos\theta)$

where P_n = n^{th} Legendre polynomial and

$$A_n = \frac{2n+1}{2}\int_0^{\pi} f(\theta)\, P_n(\cos\theta)\sin\theta\, d\theta \qquad n = 0, 1, 2, \cdots$$

$$A_0 = \frac{0+1}{2}\int_0^{\pi}(\cos^2\theta + 1)\cdot P_0(\cos\theta)\sin\theta\, d\theta \qquad P_0(x) = 1$$

$$= \frac{1}{2}\int_0^{\pi}(\cos^2\theta + 1)\sin\theta\, d\theta$$

$$= \frac{1}{2}\left(-\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi} = \frac{1}{2}\left(\frac{1}{3} + 1 - \left(-\frac{1}{3} - 1\right)\right) = \frac{1}{2}\cdot\frac{8}{3} = \boxed{\frac{4}{3}}$$

$$A_1 = \frac{2\cdot 1 + 1}{2}\int_0^{\pi}(\cos^2\theta + 1)\, P_1(\cos\theta)\sin\theta\, d\theta \qquad P_1(x) = x$$

$$= \frac{3}{2}\int_0^{\pi}(\cos^2\theta + 1)\cos\theta\sin\theta\, d\theta = \frac{3}{2}\int_0^{\pi}(\cos^3\theta + \cos\theta)\sin\theta\, d\theta$$

$$= \frac{3}{2}\left(-\frac{\cos^4\theta}{4} - \frac{\cos^2\theta}{2}\right)_0^{\pi} = \frac{3}{2}\left(-\frac{1}{4} - \frac{1}{2} - \left(-\frac{1}{4} - \frac{1}{2}\right)\right) = 0$$

$$A_2 = \frac{2\cdot 2 + 1}{2}\int_0^{\pi}(\cos^2\theta + 1)\, P_2(\cos\theta)\sin\theta\, d\theta \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$= \frac{5}{2}\int_0^{\pi}(\cos^2\theta + 1)\,\frac{1}{2}(3\cos^2\theta - 1)\sin\theta\, d\theta$$

$$= \frac{5}{4}\int_0^{\pi}(3\cos^4\theta + 2\cos^2\theta - 1)\sin\theta\, d\theta = \frac{5}{4}\left(\cancel{+}\left(\frac{3}{5}\cos^5\theta + \frac{2}{3}\cos^3\theta - \cos\theta\right)_0^{\pi}\right.$$

$$= \frac{5}{4}(-1)\left[\left(\frac{3}{5}(-1) + \frac{2}{3}(-1) + 1\right) - \left(\frac{3}{5} + \frac{2}{3} - 1\right)\right]$$

$$= \frac{5}{4}(-1)(-1)\left[2\left(\frac{3}{5} + \frac{2}{3} - 1\right)\right] = \frac{5}{4}\cdot 2\left(\frac{6}{15} + \frac{10}{15} - \frac{15}{15}\right) = \frac{5}{2}\left(\frac{1}{15}\right) = \boxed{\frac{1}{6}}$$

Then $\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) = \frac{4}{3} + \frac{1}{6} r^2 P_2(\cos\theta) + \ldots}$

§5.5 6/ $\int_{-1}^{1} P_2(x) P_3(x)\,dx = 0$ by orthogonality

7/ $\int_0^1 P_2(x)\,dx = \int_0^1 \frac{1}{2}(3x^2-1)\,dx = \frac{1}{2}\left(3\frac{x^4}{4} - x\right)\Big|_0^1 = \frac{1}{2}\left(\frac{3}{4} - 1\right)$

$$= \boxed{-\frac{1}{8}}$$

12/ $y'' - \frac{2x}{(1-x^2)} y' + \frac{6}{(1-x^2)} y = 0$ or $(1-x^2)y'' - 2xy' + 6y = 0$

Legendre's DE with $\mu = 6 = 2\cdot 3$

$n = 2$

$y = c_1 P_2(x) + c_2 Q_2(x)$ – general solution

$= c_1(3x^2-1) + c_2 Q_2(x)$,

$Q_2(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$, $a_{2k+1} = (-1)^k \frac{(n+2)(n+4)\cdots(n+2k)(n-1)(n-3)\cdots}{(2k+1)!}$

$\boxed{n=2}$

$\cdot \frac{(n-(2k-1))}{(2k+1)!} a_1$

$k=0$: $a_1 = (-1)^0 a_1 = a_1$

$k=1$: $a_3 = (-1)^1 \frac{(n+2)(n-1)}{3!} a_1$ where $n=2$

$a_3 = (-1)\frac{4\cdot 1}{3!} a_1 = -\frac{4}{6} a_1 = -\frac{2}{3} a_1$

$k=2$: $a_5 = (-1)^2 \frac{4\cdot 6\cdot 1\cdot(-1)}{5!} a_1 = \frac{\cancel{4\cdot 3\cdot 2}(-1)}{5\cdot\cancel{4\cdot 3\cdot 2}} a_1 = -\frac{1}{5} a_1$

$Q_2(x) = a_1 x - \frac{2}{3} a_1 x^3 - \frac{1}{5} x^5 + \cdots$

$= a_1\left(x - \frac{2}{3}x^3 - \frac{1}{5}x^5 + \cdots\right)$

13/ $(1-x^2)\,y'' - 2x\,y' + 6y = 0 \qquad y(0)=0,\ y'(0)=1$

$\mu = 6 = 2\cdot 3 \quad \underline{n=2}$. If there is a bounded solution, it is $y = P_2(x) = \frac{1}{2}(3x^2-1)$. But $P_2(0) \neq 0$, so there is no bounded solution in $[-1,1]$.

18/ $(1-x^2)\,y'' - 2x\,y' + 2y = 0 \qquad y(0)=1,\ y'(0)=0$

$\mu = 2 = 1\cdot 2$, so $\underline{n=1}$. The bounded solution to the DE is $P_1(x) = x$. The series solution is

$$Q_1(x) = \sum_{k=0}^{\infty} a_{2k}\, x^{2k}.$$

$k=0$: $a_0 = a_0$

$k=1$: $a_2 = -\dfrac{n(n+1)}{2!}\,a_0 = -\dfrac{1\cdot 2}{2}\,a_0 = -a_0 \qquad (n=1)$

$k=2$: $a_4 = \dfrac{(n-2)\,n\,(n+1)(n+3)}{4!}\,a_0 = \dfrac{(-1)(1)2\cdot 4}{4!}\,a_0 = -\dfrac{a_0}{3}$

$k=3$: $a_6 = -\dfrac{(n-4)(n-2)\,n\,(n+1)(n+3)(n+5)}{6!}\,a_0 = -\dfrac{(-3)(-1)\,1\cdot 2\cdot 4\cdot 6}{6!}\,a_0$

$$= -\frac{4\cdot 3\cdot 2\cdot 1\cdot 6}{6!}\,a_0 = -\frac{a_0}{5}$$

$$\boxed{Q_1(x) = a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 + \cdots\right)}$$