Solutions - Homework #1

§1.1 #6 §1.2 #6, 8, 10

1.1 #6. Find general solution by appropriate change of variables:

$$2\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$$

Let $\alpha = ax + bt$, $\beta = cx + dt$. Then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}$$

$$2\left(b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}\right) + 3\left(a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta}\right) = 0$$

$$(2b+3a)\frac{\partial u}{\partial \alpha} + (2d+3c)\frac{\partial u}{\partial \beta} = 0$$

Let $a=2$, $b=-3$, $c=1$, $d=0$. Then

$$\overset{0}{\cancel{(6-6)}}\frac{\partial u}{\partial \alpha} + 3\frac{\partial u}{\partial \beta} = 0, \text{ or } \frac{\partial u}{\partial \beta} = 0. \text{ Hence}$$

$$u = f(\alpha) = f(2x-3t), \quad f \text{ differentiable.}$$

1.2 #6. Let $u(x,t) = F(x+ct) + G(x-ct)$, F, G diff, F', G' diff.
Then u solves the wave equation $(*)$ $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.

Solve for u if $u(x,0) = e^{-x^2}$, $\frac{\partial u}{\partial t}(x,0) = 0$ $\forall x \in \mathbb{R}$.

$$\frac{\partial u}{\partial t} = cF'(x+ct) - cG'(x-ct)$$

$$\begin{cases} u(x,0) = F(x) + G(x) = e^{-x^2} \\ \frac{\partial u}{\partial t}(x,0) = cF'(x) - cG'(x) = 0 \quad \text{or} \quad F'(x) = G'(x) \end{cases}$$

Then $G(x) = F(x) + K$, $K \in \mathbb{R}$.

$\Rightarrow$ $F(x) + G(x) = 2F(x) = e^{-x^2}$ or $F(x) = \frac{1}{2}e^{-x^2}$.

$$u(x,t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x-ct)^2}$$

8. Same as #6 with $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=\frac{x}{(1+x^2)^2}$ $\quad \forall x\in\mathbb{R}$

$$u(x,0)= F(x)+G(x)=0 \quad \text{or} \quad G(x)=-F(x) \Rightarrow G'=-F'$$

$$\frac{\partial u}{\partial t}(x,0)= cF'(x)-cG'(x) = \frac{x}{(1+x^2)^2}$$

Then $\quad c(F'(x)+F'(x)) = \frac{x}{(1+x^2)^2}$, or

$$F'(x)= \frac{1}{2c}\frac{x}{(1+x^2)^2} \Rightarrow F(x)=\frac{1}{2c}\cdot\frac{1}{2}\cdot\frac{1}{(1+x^2)}+K$$

$$F(x)=\frac{1}{4c}\frac{1}{1+x^2} = -G(x)$$

$$u(x,t)=\frac{1}{4c}\left[\frac{1}{1+(x+ct)^2}-\frac{1}{1+(x-ct)^2}\right]$$

10. Use #6, #8 to solve wave equation with

$$u(x,0) = e^{-x^2}=f(x)$$

$$\frac{\partial u}{\partial t}(x,0)=\frac{x}{(1+x^2)^2}=g(x), \quad x\in\mathbb{R}.$$

From #6, $u_1(x,t)=\frac{1}{2}\left(e^{-(x+ct)^2}+e^{-(x-ct)^2}\right)$ solves

$$u_1(x,0)=f, \quad \frac{\partial u_1}{\partial t}(x,0)=0.$$

From #8, $u_2(x,t)=\frac{1}{4c}\left[\frac{1}{1+(x+ct)^2}-\frac{1}{1+(x-ct)^2}\right]$ solves

$$u_2(x,0)=0, \quad \frac{\partial u_2}{\partial t}(x,0)=g(x).$$

Then $\quad u=u_1+u_2$

$$=\frac{1}{2}\left(e^{-(x+ct)^2}+e^{-(x-ct)^2}\right)+\frac{1}{4c}\left(\frac{1}{1+(x+ct)^2}-\frac{1}{1+(x-ct)^2}\right)$$

solves $u(x,0)=f+0=f(x)$

$$\frac{\partial u}{\partial t}(x,0) = 0+g = g(x).$$