Review 2/12/08

Note Title 2/12/2008

Highlights

Ch. 1.

- Method of change of variables for linear homogeneous PDE.

  Linear change of variables

  $$\left.\begin{aligned} \alpha &= ax+bt \\ \beta &= cx+dt \end{aligned}\right\}$$

- Method of separation of variables (best example - wave equation)

— "Method of superposition"

For linear homogeneous PDE's, if u_1, u_2 are solutions, then $u = c_1 u_1 + c_2 u_2$, $c_1, c_2 \in \mathbb{R}$, is also a solution.

Chapter 2. Fourier series

— 2p periodic function — find a_0, a_n, b_n for Fourier series

— Theorem If f is continuous and $\left\{ \begin{array}{l} C^1, \\ \text{or} \\ \text{piecewise-smooth} \end{array} \right.$ and 2p-periodic then f(x) is represented by its Fourier series.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right).$$

- Changes of variable to find Fourier series given the Fourier series for a simpler function.

- Half Range Expansions - Sine series
  Cosine series

- Mean Square Error -

Defn $E_N = \frac{1}{2p}\int_{-p}^{p} (f - S_N)^2$ S_N = partial sum of Fourier Series.

<u>Theorem</u> $E_N = \frac{1}{2p}\int_{-p}^{p} f^2 - a_0^2 - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2)$

Find Fourier coefficients for f
use Theorem to find E_N.

Parseval's Identity

$$\frac{1}{2p}\int_{-p}^{p} f^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

- Vibrating String with External Force $F(t)$

Theorem for solution in terms of Fourier series for $F(t)$

(I will give you formulas for α_n, β_n.)

- Pointwise convergence of sequence of function on $E \subseteq \mathbb{R}$

Defn

- uniform convergence of sequence (f_n) on E

defn

calculation

find ~~limit~~ f by ptwise convergence.

- ptwise, uniform convergence of series $\sum_{1}^{\infty} u_n$ on E

- Weierstrass M-test.

- (f_n), f_n continuous on E $\forall n$, $f_n \to f$ uniformly on E $\Rightarrow$ f continuous on E.

  [f not cont. at some $x_0 \in E \Rightarrow f_n \not\to f$ uniformly on E.

- (f_n) as above, $E = [a,b]$ $\Rightarrow$

$$\lim_n \int_a^b f_n = \int_a^b \lim f_n = \int_a^b f$$

— (f_n) as above, $E=[a,b]$, (f_n') converges uniformly on E $\Rightarrow$ $\lim_n f_n' = f' = \frac{d}{dx}(\lim_n f_n)$

$$f_n = \frac{\sin nx}{\sqrt{n}} \quad \text{on } [0, 2\pi].$$

Find pointwise limit.

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad |\sin nx| \le 1$$

$$\frac{1}{\sqrt{n}} \to 0$$

$$0 \le \left| f_n(x) - 0 \right| = \left| \frac{\sin nx}{\sqrt{n}} - 0 \right| = \left| \frac{\sin nx}{\sqrt{n}} \right| = \frac{|\sin nx|}{\sqrt{n}}$$

$$\le \frac{1}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty$$

Then $f_n(x) \to 0$ for each $x \in [0, 2\pi]$

Is the convergence uniform?

$$|f_n(x) - f(x)| = |f_n(x) - 0| = \left|\frac{\sin nx}{\sqrt{n}}\right| = \frac{|\sin nx|}{\sqrt{n}} \le \frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

upper bound for $|f_n(x) - f(x)|$

$f_n \to 0$ uniformly on $[0, 2\pi]$.

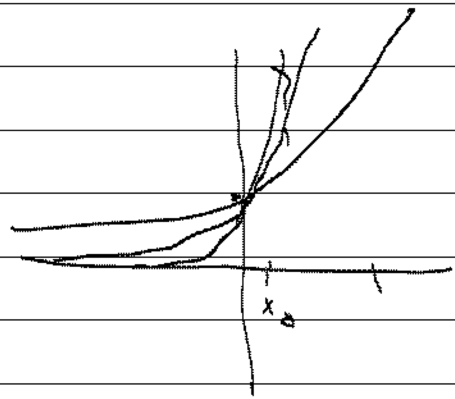
4/ $f_n(x) = e^{-nx}$ on $[0,1]$

$$f_n(x) = e^{-nx} = \begin{cases} 1 & x = 0 \\ e^{-nx} & 0 < x \le 1 \end{cases} \longrightarrow \begin{cases} 1 & x = 0 \\ 0 & 0 < x \le 1 \end{cases}$$

Hence $f_n \not\rightarrow f$ uniformly on $[0,1]$. Not continuous on $[0,1]$

Alternate proof on uniform convergence:

$$|f_n(x) - f(x)| = |e^{-nx} - f(x)| = \begin{cases} 1-1 & x=0 \\ e^{-nx} & x>0 \end{cases}$$



as $x \to 0$

e^{-nx} gets larger

$$\sum \frac{x^n}{9^n} \qquad |x| \le 7 \qquad \left|\frac{x^n}{9^n}\right| \le \frac{7}{9} < 1$$

Converges unif. on $\mathbb{R}$ by Weierstrass M-test (geometric series, ratio $= \frac{7}{9}$)

Term-by-Term Differentiation.

$f(x) = |x|$ on $[-\pi, \pi]$

Fourier series for f is $\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x$

~~f is continuous~~

$$f'(x) = \begin{cases} 1 & \text{on } (0, \pi] \\ -1 & \text{on } [-\pi, 0) \\ DNE & x = 0 \end{cases}$$

f' is continuous except for $x = 0$, where $f'(0)$ does not exist (DNE).

$$|x| = f(x) = \frac{\pi}{2} - \frac{4}{\pi} \boxed{\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x} \quad (1) \quad \text{on } [-\pi, \pi] \; \cancel{\text{ext}}$$

~~for~~

$a_0 = \frac{\pi}{2}, \quad a_{2k+1} = -\frac{4}{\pi} \frac{1}{(2k+1)^2} \quad b_n = 0$

Take term-by-term derivative:

$$f' = 0 - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{-1}{(2k+1)^{\cancel{2}}} \sin(2k+1)x \; (2k+1)$$

$$= + \frac{4}{\pi} \boxed{\sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x} \quad (2)$$

$a_0 = a_n = 0$

$b_{2k+1} = \frac{4}{\pi} \frac{1}{2k+1}$

Does this equal $f'(x)$ for $x \neq 0$?

~~Thm~~ If (1) and (2) converge uniformly on $[-\pi, \pi]$,

then $f'(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)x$

The theorem will not apply since (2) does not

Converge uniformly.