Solutions
NAME

IE/MA/OR 505 EXAM II

1	2	3	4	t
21	31	27	21	100

1. (21) Given that the simplex method leads from the following input tableau M_0 (for which $x \geq 0$) to the following terminal tableau M_t:

$$M_0 = \begin{array}{c} \begin{array}{cccc} & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{c|ccc} -1 & 1 & 1 & -1 \\ \hline 1 & -2 & 2 & 1 \\ -1 & 3 & -1 & -1 \end{array}\right] \end{array} \qquad M_t = \begin{array}{c} \begin{array}{cccc} & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{c|ccc} 0 & 0 & 4 & 0 \\ \hline 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

$T = (3, 1) \Rightarrow T' = (2)$

(a) determine the matrix Q_{t0} for which $M_t = Q_{t0} M_0$

$$Q_{t0} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ because}$$

Since $Q_{0t} M_t = M_0$ and hence $Q_{0t} = [u\ M_0^T] = \begin{bmatrix} 1 & c^T \\ 0 & A^T \end{bmatrix}$, $Q_{0t}^{-1} = \begin{bmatrix} 1 & -c^T[A^T]^{-1} \\ 0 & [A^T]^{-1} \end{bmatrix} = Q_{t0}$

and hence $M_t = \left[\begin{array}{c|c} d - c^T[A^T]^{-1}b & c - c^T[A^T]^{-1}A \\ \hline [A^T]^{-1}b & [A^T]^{-1}A \end{array}\right]$, which implies that

$f^* = -d + c^T[A^T]^{-1}b$, $x^{*T} = [A^T]^{-1}b$, $x^{*T'} = 0$ provided $\begin{cases} [A^T]^{-1}b \geq 0 \\ c^{T'} - c^T[A^T]^{-1}A^{T'} \geq 0 \end{cases}$

(b) do a post-optimality analysis assuming that the input constraint coefficient matrix and terminal basic sequence remain fixed -- by giving explicit formulas in terms of b_1, b_2, c_1, c_2, c_3 and d for:

Since $[A^T]^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ by matrix inversion and $[A^T]^{-1}A^{T'} = A_t^{T'} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$f^* =$ $\qquad x^*_1 =$ $\qquad x^*_2 =$ $\qquad x^*_3 =$

$f^* = -d + c_1(b_1 + b_2) + c_3(3b_1 + 2b_2)$

$x_1^* = b_1 + b_2$ $\quad x_2^* = 0$ $\quad x_3^* = 3b_1 + 2b_2$

which are valid provided that

feasibility $\begin{Bmatrix} b_1 + b_2 \geq 0 \\ 3b_1 + 2b_2 \geq 0 \end{Bmatrix} \Rightarrow$ ranges $\begin{cases} b_1 \geq 1 \\ b_2 \geq -1 \end{cases}$

optimality $\{-c_1 + c_2 - 4c_3 \geq 0\} \Rightarrow$ ranges $\begin{cases} c_1 \leq 5 \\ c_2 \geq -3 \\ c_3 \leq 0 \end{cases}$

(c) determine each of the following partial derivatives that definitely exist, and mark with an × those that might not exist

$f^*_{b_1} =$ X $\qquad f^*_{b_2} =$ X $\qquad f^*_{c_1} = 0$ $\qquad f^*_{c_2} = 0$ $\qquad f^*_{c_3} = 1$ $\qquad f^*_d = -1$

because $b_1 = 1$ is on the boundary of $b_1 \geq 1$

because $b_2 = -1$ is on the boundary of $b_2 \geq -1$

2. (31) Perform one iteration of the dual simplex algorithm on each of the following canonical schemas (either by giving an optimal value and solution, or by saying why there are no feasible solutions, or by producing a single new schema):

		x_1	η_1	
	4	2	1	$-f$(min)
ξ_2	3	-1	1	x_2
y_2	1	0	2	η_2
		ξ_1	y_1	

$f^* = -4$
$x_1^* = 0$
$x_2^* = 3$
$\eta_1^* = 0$
$\eta_2^* = 1$

		x_1	η_1	
	4	1	2	$-f$(min)
ξ_2	3	-1	1	x_2
y_2	-1	0	2	η_2
	$-g$	ξ_1	y_1	

There are no feasible solutions because constraint 2, namely $-1 = 0x_1 + 2\eta_1 + \eta_2$ can not be satisfied for $x \geq 0$ and $\eta \geq 0$

		x_1	η_1	
	2	1	2	$-f$(min)
ξ_2	-3	-1	-1	x_2
y_2	2	1	-1	η_2
	$-g$	ξ_1	y_1	

	x_2	η_1	
-1	1	1	$-f$
3	-1	1	x_1
-1	1	-2	η_2

Also, insert the appropriate dual labels in one of the three preceding given schemas (your choice)

3. (27) Given the following non-canonical primal-dual schema where x_2 and η_2 are restricted, but x_1 is unrestricted and η_1 is artificial

		x_1	x_2	
	0	-1	2	$-f$(min)
y_1	2	1	0	η_1
y_2	1	-1	1	η_2
	$-g$(min)	ξ_1	ξ_2	

(a) classify y_1, y_2, ξ_1, ξ_2 as restricted, unrestricted or artificial:

y_1 is unrestricted y_2 is restricted ξ_1 is artificial ξ_2 is restricted

(b) state the appropriate complementary slackness conditions (in their simplest form)

$x_2 = 0$ or $\xi_2 = 0$, $y_2 = 0$ or $\eta_2 = 0$

(c) reduce the given schema to canonical form, including a statement of any equations that must be stored to eventually determine complete primal and dual optimal solutions.

		η_1	x_2	
	2	1	2	$-f$
~~ξ_1~~	~~2~~	~~1~~	~~0~~	~~x_1~~
y_2	3	1	1	η_2
	$-g$	y_1	ξ_2	

Canonical form

		x_2	
	2	2	$-f$
y_2	3	1	η_2
	$-g$	ξ_2	

with stored equations

$2 = 0x_2 + x_1$
$1 = -1y_2 + y_1$
that determine x_1^* and y_1^* from x_2^* and y_2^*

4. (21) Given the following pair of primal and dual linear optimization problems P and Q

Problem P: Minimize $0x$ subject to
$Ax = b$
x unrestricted

Problem Q: Minimize yb subject to
$-yA = 0$
y unrestricted

(a) which, if any, of these two problems P and Q are always consistent (for all compatible matrices A, b, and 0)

Problem Q, because $y = 0$ is clearly feasible

(b) which, if any, of these two problems P and Q are always bounded when consistent

Problem P, because Problem Q is always consistent

(c) since it is known from linear algebra that each "vector space" V can be represented as the "column space" for some matrix A (ie, as the set $\{Ax \mid x \text{ is unrestricted}\}$), and since it is also known that its orthogonal complement $V^{\perp} = \{\text{unrestricted } z \mid zA = 0\}$, parts (a) and (b) along with our duality theory imply that problem Q is bounded if, and only if, b has what relation to V?

b is in V