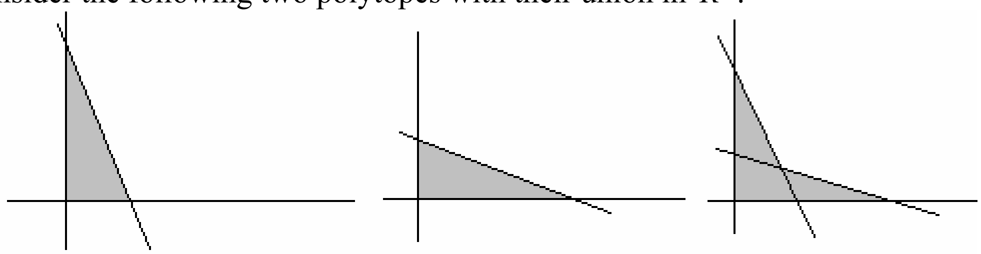
Exercise for Part X **(Px means exercise on page x)**

P8 ex1 (a) The intersect of P_{A1}^{b1}, P_{A2}^{b2} is given by $P_{A1}^{b1} \cap P_{A2}^{b2} = \{x \in R^n \mid \begin{bmatrix} A1 \\ A2 \end{bmatrix} x \le \begin{bmatrix} b1 \\ b2 \end{bmatrix}\}$. If A1 and A2 do not have the same number of columns, i.e., A1 has m columns and A2 has n columns where m<n, just add n-m zero columns to A1.

(b) The proof is similar to (a). Since each polytope is bounded, the intersect of two polytopes must be bounded, hence form a new polytope.

P8 ex2 Example. $P_A^b = \{x \in R^n \mid Ax \le b\}, P_A'^b = \{x \in R^n \mid Ax \ge b\}$, the union of these two is R^n, which is not a polyhedral.

Polytope is a convex set. But the union of two convex sets is not necessarily convex. Consider the following two polytopes with their union in R^2.



P8 ex3 The complement of a polyhedral is an open set, thus can not be a polyhedral.

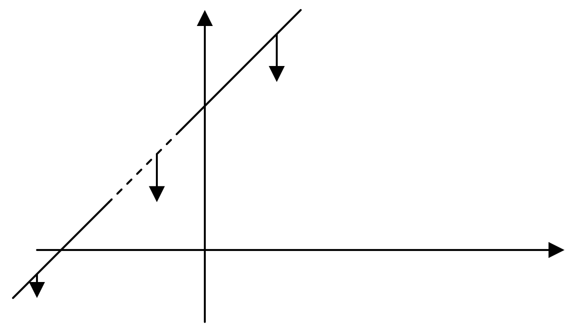
Suppose the complement of a polytope is bounded, then their union should also be bounded. But there union is R^n, which is not bounded. Thus the complement of a polytope is not bounded, hence is not a polytope.

P10 ex1 For $x_1, x_2 \in H_a^b$, we have $a(\delta_1 x_1 + \delta_2 x_2) = \delta_1 a x_1 + \delta_2 a x_2 = (\delta_1 + \delta_2) b = b$. So $\delta_1 x_1 + \delta_2 x_2 \in H_a^b$.

P10 ex2 If $\delta_1, \delta_2 \ge 0$, we have $a(\delta_1 x_1 + \delta_2 x_2) = \delta_1 a x_1 + \delta_2 a x_2 \le (\delta_1 + \delta_2) b = b$. So $\delta_1 x_1 + \delta_2 x_2 \in P_a^b$ .But if δ_1 or δ_2 is less than zero, then the $\le$ sign does not hold. For example, $P_1^0 = \{x \mid x \le 0\}$, we choose $x_1 = -1, x_2 = 0, \delta_1 = -1, \delta_2 = 2$, then $\delta_1 x_1 + \delta_2 x_2 = 1$, which does not belong to the halfspace.

P10 ex3 Change the $\le$ sign above to <.

P10 ex4

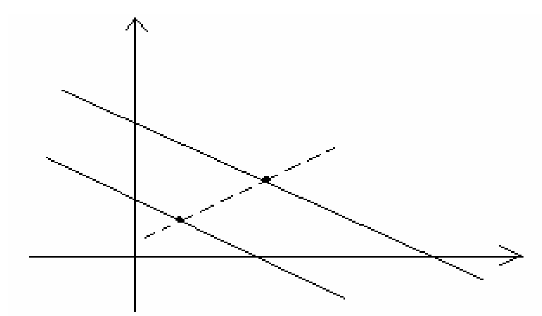


A half line must be convex. Since here we have only one dimension, the half line must be closed or open, which is convex in either case.

P13 ex1 The set of all affine combinations of two identical points is just that point. The set of all affine combinations of three collinear but distinct points is the line which lines up these three points. The set of all affine combinations of four planar but noncollinear points in R^3 is the plane containing these points.
Similarly, the set of all convex combinations of two identical points is just that point. The set of all convex combinations of three collinear but distinct points is the line segment between the two points which have the longest distance. The set of all convex combinations of four planar but noncollinear points in R^3 is the rectangular whose vertexes are these four points.

P16 ex1 (a) Given two convex sets P_1, P_2 and their intersect $P_1 \cap P_2$. If it is empty, then the proposition is proved. Else, if it has only one element, then the proposition is still true. Now suppose it has two or more elements, $x_1, x_2 \in P_1 \cap P_2$, then $x_1, x_2 \in P_1$ and $x_1, x_2 \in P_2$. Since P_1, P_2 are convex, then for any $x = \delta x_1 + (1-\delta)x_2, 0 \le \delta \le 1$, it must belongs to P_1, P_2, and thus belongs to $P_1 \cap P_2$, which means $P_1 \cap P_2$ is convex. Use induction to prove the case with more convex sets.
(b) Quite similar to (a)

P16 ex2 (a) Example. $P_1 = \{x \in R \mid x \le 0\}$, $P_2 = \{x \in R \mid x \ge 1\}$, then the points between 0 and 1 does not belong to $P_1 \cup P_2$.
(b) Example.



P_1, P_2 are two distinct lines. Pick one point from each line, line up these two points, then any point on this line (different from that we picked initially) does not belong to $P_1 \cup P_2$.

P16 ex3 Consider the complement of the convex set $P = \{(x_1, x_2) \in R^2 \mid {x_1}^2 + x_2^2 \le 1\}$.

The affine set is given by $P = \{x \in R^n \mid Ax = b\}$, its complement is given by $\overline{P} = \{x \in R^n \mid Ax \ne b\}$. Suppose it is affine. For x_1, x_2 such that $Ax_1 = b - 1, Ax_2 = b + 1$, then it is obvious that $x_1, x_2 \in \overline{P}$. But $x = 0.5x_1 + 0.5x_2$ does not belong to $\overline{P}$, since in this case $Ax = b$. This contradicts with the definition of affine set.

P17 ex1 (a) The cones are $\{x \in R \mid x \le 0\}$, $\{x \in R \mid x \ge 0\}$, $\{x \in R \mid x = 0\}$, $\{x \in R\}$.

(b) $\{x \in R\}$

(c) $\{x \in R \mid x \le 0\}$, $\{x \in R \mid x \ge 0\}$, $\{x \in R \mid x = 0\}$

(d) $\{x \in R \mid x \le 0\}$, $\{x \in R \mid x \ge 0\}$, $\{x \in R \mid x = 0\}$

(e) $\{x \in R \mid x = 0\}$

(f) $\{x \in R\}$

(g) all

P17 ex2 (a) Given two cones C_1, C_2, for any $x \in C_1 \cap C_2$, and any $s \ge 0$, we have $sx \in C_1$ since $x \in C_1$, and $sx \in C_2$ since $x \in C_2$, hence $sx \in C_1 \cap C_2$, which means the intersect is a cone (It contains at least the element 0, hence not empty).

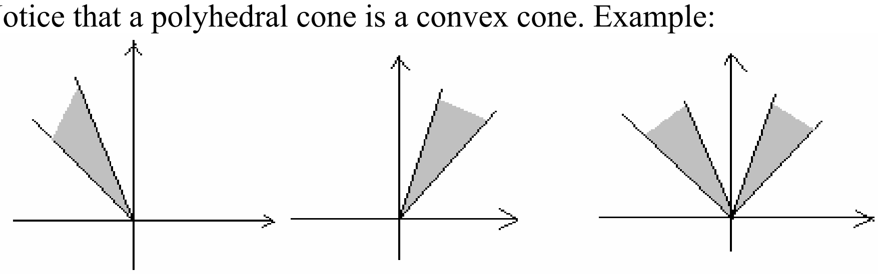
(b) Given two cones C_1, C_2, for any $x \in C_1 \cup C_2$, suppose $x \in C_1$, then $sx \in C_1$ and thus $sx \in C_1 \cup C_2$. So the union is also a cone.

(c) It does not contain 0.

P17 ex 3 (a) A polyhedral cone is given by $\{x \in R^n \mid Ax \le 0\}$. Its intersect with another polyhedral cone $\{x \in R^n \mid Bx \le 0\}$ is given by $\{x \in R^n \mid \begin{bmatrix} A \\ B \end{bmatrix} x \le 0\}$, which is also a polyhedral cone. 0 is clearly a feasible element, thus the intersect is not empty. Use induction to finish the proof.

(b) The intersect is a cone, which was proved in exercise 2. The intersect of two convex sets is convex, which was proved in P16 exercise 1.

P18 ex4 Notice that a polyhedral cone is a convex cone. Example:



Since the union is not convex, it can not be a polyhedral cone either.

P18 ex5 $\{0\}$

P18 ex6 An affine set is given by $\{x \mid Ax = b\}$. In order for it to be a cone, it must contain 0, which implies b=0. The converse is obvious.

P18 ex7 A polyhedral is $P = \{x \mid Ax \leq b\}$. In order for it to be a cone, it must contain 0, from which we get $0 \leq b$. Suppose $b_i > 0$, which is the i th element of b, then consider the i th row of $Ax \leq b$, which is $a_i x \leq b_i$. Suppose there exists $x' \in P$ such that $a_i x' > 0$, then because P is a cone, we have $sx' \in P, \forall s \geq 0$. That means $a_i sx' \leq b_i$. But because $a_i x' > 0$, we can choose s large enough so that $a_i sx' > b_i$. This is a contradiction. Therefore, we must have $a_i x \leq 0, \forall x \in P$. That is, if we let $b_i = 0$, then this operation does not alter the polyhedral set. Thus we can have b=0. The converse is obvious.

P18 ex8 $P = \{x \mid Ax \leq 0\}$ is pointed $\Rightarrow$ $\{x \mid Ax = 0\} = \{0\}$

Suppose it is not true, and we have $x' \neq 0, x' \in P$ such that $Ax' = 0$, then $A(-x') = 0$, hence $-x' \in P, -x' \neq 0$. This is a contradiction since P is pointed.

$\{x \mid Ax = 0\} = \{0\} \Rightarrow P = \{x \mid Ax \leq 0\}$ is pointed

Suppose it it not pointed, then there exist $x' \neq 0, -x' \neq 0, x' \in P, -x' \in P$ such that $Ax' \leq 0, A(-x') \leq 0$ or equivalently $Ax' \leq 0, Ax' \geq 0 \Rightarrow Ax' = 0, x' \neq 0$. This is also a contradiction with $\{x \mid Ax = 0\} = \{0\}$, so P must be pointed.