Solutions
NAME

MA/OR 505 EXAM II

1	2	3	4	5	6	t
5	25	18	12	20	20	100

1. Reformulate the problem: Maximize $3x_1 + 2x_2 - 4x_3 + 2$ subject to
(5) in a form to which our duality applies

$2x_1 - x_2 + 3x_3 \le -2$

$-x_1 + x_2 + 2x_3 \ge -1$

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

Minimize $-3x_1 - 2x_2 + 4x_3 - 2$
subject to $2x_1 - x_2 + 3x_3 \le -2$
$x_1 - x_2 - 2x_3 \le 1$
$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

2. Given the problem: Minimize $-2x_1 + 4x_2 - 3x_3 + 1$ subject to
(25)

$-3x_1 - x_2 + 2x_3 \le -2$

$-x_1 + x_2 - 4x_3 \le 1$

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

(a) state it's dual problem (without slack variables)
(5)

Minimize $-2y_1 + y_2 - 1$ subject to
$3y_1 + y_2 \le -2$
$y_1 - y_2 \le 4$
$-2y_1 + 4y_2 \le -3$
$y_1 \ge 0, y_2 \ge 0$

(b) set up a primal-dual schema representing both problems
(5)

		x_1	x_2	x_3	
	-1	-2	4	-3	$-f$ (min)
y_1	-2	-3	-1	2	η_1
y_2	1	-1	1	-4	η_2
	$-g$ (min)	ξ_1	ξ_2	ξ_3	

(c) state the appropriate complementary slackness conditions
(5)

$x_1\xi_1 = 0,\ x_2\xi_2 = 0,\ x_3\xi_3 = 0;\ y_1\eta_1 = 0,\ y_2\eta_2 = 0$

If x_2 becomes unrestricted and the inequality constraint $-x_1 + x_2 - 4x_3 \le 1$ becomes an equality constraint

(d) how does the answer to part (a) change?

$y_1 - y_2 \le 4$ is replaced by $y_1 - y_2 = 4$, and y_2 becomes unrestricted.

(5) (e) how does the answer to part (c) change?

$x_2\xi_2 = 0$ and $y_2\eta_2 = 0$ are deleted from the list.

3. (18) Perform one <u>complete</u> iteration of the <u>dual</u> simplex algorithm on each of the following schemas:

(6)

	η_2	x_2	x_3	
–4	3	1	2	–f(min)
–2	(–1)	0	2	η_1
1	–1	–2	–3	x_1

	η_1	x_2	x_3	
–10	3	1	8	–f(min)
2	–1	0	–2	η_2
3	–1	–2	–5	x_1

(6)

	η_2	x_2	x_3	
–2	3	2	1	–f(min)
1	2	–1	0	η_1
2	–2	1	–4	x_1

$x_1^* = 2$
$x_2^* = 0$
$x_3^* = 0$ $f^* = 2$
$\eta_1^* = 1$
$\eta_2^* = 0$

(6)

	η_2	x_2	x_3	
4	3	1	2	–f(min)
2	1	0	2	η_1
–1	1	2	3	x_1

Problem is inconsistent because constraint $-1 = \eta_2 + 2x_2 + 3x_3 + x_1$ can not be satisfied for $\eta_2 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 \geq 0$

4. (12) Given the terminal primal–dual schema

		x_3	η_1	η_2	
	0	3	1	0	–f(min)
ξ_1	1	3	2	1	x_1
ξ_2	0	1	1	1	x_2
	–g(min)	ξ_3	y_1	y_2	

(4) (a) insert the dual labels

(4) (b) state both a primal optimal solution and a dual optimal solution

$f^* = 0, x_1^* = 1, x_2^* = 0, x_3^* = 0, \eta_1^* = 0, \eta_2^* = 0$

$g^* = 0, \xi_1^* = 0, \xi_2^* = 0, \xi_3^* = 3, y_1^* = 1, y_2^* = 0$

(4) (c) which, if any, of the optimal values given in part (b) must be unique?

$\xi_1^* = 0, x_3^* = 0, \eta_1^* = 0$ by complementary slackness

5. (20) Given a vector b and a matrix A where b is <u>not</u> in the cone $\{Ax \mid x \geq 0\}$, answer each of the following questions (including a reason for each "yes" or "no"):

(5) (a) is the linear optimization problem

Min 0x subject to
$Ax = b$
$x \geq 0$

consistent?

No, because b is not in the cone $\{Ax \mid x \geq 0\}$

(5) (b) what is the dual of this problem?

Minimize yb subject to
$-yA \leq 0$
y unrestricted

(5) (c) is there a dual feasible solution with a zero dual objective value?

Yes, namely $y = 0$

(5) (d) if your answer to part (c) was "yes", is there also a dual feasible solution with a negative dual objective value?

Yes, because the primal problem is inconsistent while the dual problem is consistent and hence unbounded

6. (20) Given the following input tableau M_0 and terminal tableau M_t for a particular application of the simplex method

$$M_0 = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline -1 & 1 & 1 & -1 & 0 & 0 \\ 2 & -2 & 2 & 1 & 1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 1 \end{array} \qquad \mathcal{O} = (4,5)$$

$$M_t = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 2 & 0 & 4 & 0 & 2 & 1 \\ 4 & 0 & 4 & 1 & 3 & 2 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{array} \qquad \mathcal{J} = (3,1)$$

(a) (2) determine:

$[A^{\mathcal{J}}]^{-1} = A_t^{\mathcal{O}} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\qquad [A^{\mathcal{J}}]^{-1}[A^{\mathcal{J}'}] = A_t^{\mathcal{J}'} = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(b) (11) for fixed A and $\mathcal{J}$, describe the dependence of f^* and x^* on b, c, and d:

$f^* = -d + c^{\mathcal{J}}[A^{\mathcal{J}}]^{-1}b = -d + (c_3\ c_1)\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$= -d + 3b_1c_3 + 2b_2c_3 + b_1c_1 + b_2c_1$

$\begin{pmatrix} x_3^* \\ x_1^* \end{pmatrix} = x^{*\mathcal{J}} = [A^{\mathcal{J}}]^{-1}b$ and $\begin{pmatrix} x_2^* \\ x_4^* \\ x_5^* \end{pmatrix} = x^{*\mathcal{J}'} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; so

$x_1^* = b_1 + b_2$ $\qquad x_2^* = 0$ $\qquad x_3^* = 3b_1 + 2b_2$

$x_4^* = 0$ $\qquad x_5^* = 0$

provided that $b_1 + b_2 \geq 0$, $3b_1 + 2b_2 \geq 0$, and

$-c_1 + c_2 - 4c_3 \geq 0$, $-c_1 - 3c_3 + c_4 \geq 0$, $-c_1 - 2c_3 + c_5 \geq 0$

where the last three inequalities come from

$c^{\mathcal{J}'} - c^{\mathcal{J}}[A^{\mathcal{J}}]^{-1}A^{\mathcal{J}'} = (c_2\ c_4\ c_5) - (c_3\ c_1)\begin{bmatrix} 4 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \geq (0\ 0\ 0)$

(c) (7) determine each of the following partial derivatives that exist, and mark with a × those that are nonexistent:

$f^*_{b_1} = -2$ $\quad f^*_{b_2} = -1$ $\quad f^*_{c_1} = 1$ $\quad f^*_{c_2} = 0$ $\quad f^*_{c_3} = 4$ $\quad f^*_{c_4} = 0$ $\quad f^*_{c_5} = 0$

because $\nabla_b f^* = -c_t^{\mathcal{O}} = (-2, -1)$

$\nabla_{c^{\mathcal{J}}} f^* = b_t = (4\ 1)$

$\nabla_{c^{\mathcal{J}'}} f^* = (0, 0, 0)$

when $b_t > 0$ and $c_t^{\mathcal{J}'} > 0$ (which is the case)