MA/OR 505 - Exam I

Name: Solutions

1	2	3	4	t
42	35	8	15	100

Notes: Circled numbers give the point allocation. If you can not do the required work in the space provided, use the back of these sheets (with a note saying which ones).

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1. Perform one complete iteration of the simplex algorithm on:

(a) each of the following schemas

		x_3	x_1	
	2	2	1	$-f$(min)
	3	2	-2	x_2
	2	-1	1	x_4

Terminates on step 1 with optimal solution $x^* = (0,3,0,2)$ and optimal value $f^* = -2$

		x_1	x_3	
	4	-2	1	$-f$(max)
	0	-1	1	x_2
	1	0	2	x_4

		x_1	x_2	
	4	-1	-1	$-f$(max)
	0	-1	1	x_3
	1	2	-2	x_4

(b) each of the following tableaus

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{c|ccccc} & \downarrow & & & & \\ & x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\quad -f(\max)$$

$$\left[\begin{array}{c|ccccc} 4 & 2 & 0 & -1 & 0 & 0 \\ \hline 2 & -2 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 3 & -1 & 1 & 1 & 0 & 0 \end{array}\right] \quad \mathcal{S} = (5,4,2)$$

Terminates on step 2 with $x_1 \to +\infty$ and hence

$x_2 = 3 + x_1 \to +\infty$

$x_3 = 0$

$x_4 = 1$

$x_5 = 2 + 2x_1 \to +\infty$

$f(x) = -4 + 2x_1 \to +\infty$

$$\begin{array}{ccccc} & & & \downarrow & \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\quad -f(\min)$$

$$\left[\begin{array}{c|ccccc} 2 & 0 & 0 & 0 & -1 & 1 \\ \hline 0 & 0 & 0 & 1 & 2 & -2 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 1 \end{array}\right] \leftarrow \quad \mathcal{S} = (3,2,1)$$

$$\left[\begin{array}{c|ccccc} 2 & 0 & 0 & 1/2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/2 & 1 & -1 \\ 2 & 0 & 1 & -1/2 & 0 & 1 \\ 1 & 1 & 0 & 1/2 & 0 & 0 \end{array}\right] \quad \mathcal{S} = (4,2,1)$$

Transform:

(c) the first schema given in part (a) into an equivalent tableau.

$$\begin{array}{c} \\ \end{array}\begin{array}{cccc} x_3 & x_1 & x_2 & x_4 \end{array}$$

$$\left[\begin{array}{c|cccc} 2 & 2 & 1 & 0 & 0 \\ \hline 3 & 2 & -2 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left[\begin{array}{c|cccc} 2 & 1 & 0 & 2 & 0 \\ \hline 3 & -2 & 1 & 2 & 0 \\ 2 & 1 & 0 & -1 & 1 \end{array}\right] \quad \mathcal{S} = (2,4)$$

(d) the first tableau given in part (b) into an equivalent schema.

$$\begin{array}{c|cc|l} & x_1 & x_3 & \\ \hline 4 & 2 & -1 & -f(\max) \\ \hline 2 & -2 & 3 & x_5 \\ 1 & 0 & 2 & x_4 \\ 3 & -1 & 1 & x_2 \\ \hline \end{array}$$

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2. Given the (unlabeled) schema

d	c
b	A

(a) state a condition involving only this schema that implies no feasible solutions exist (for the corresponding optimization problem),

Some distinguished column element $b_k < 0$ while $a_{kj} \geq 0$ for each j

(b) state a condition involving only the distinguished column b that implies a feasible solution does exist,

$b \geq 0$

(c) state a condition involving only the distinguished column b that guarantees the next pivot (if any) in phase II will be nondegenerate

$b > 0$

(d) if the condition stated in part (c) is not satisfied, does the next pivot have to be degenerate?

No

(e) under what condition is it impossible to pivot on an element a_{ij} in A (whether in phase I or phase II)?

$a_{ij} = 0$

3. Assuming that the linear optimization problem

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$$\begin{aligned} \text{Minimize} \quad & (cD)y \\ \text{subject to} \quad & (AD)y = b \\ & y \geq 0 \end{aligned}$$

has an optimal solution y* and that D is a "diagonal matrix" with positive elements on its diagonal, give in terms of the optimal solution y* a formula for the optimal solution x* to the related linear optimization problem

$$\begin{aligned} \text{Minimize} \quad & cx \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

[Hint: Find a "mapping" from y to x that shows that these two optimization problems are "equivalent".]

The mapping $x = Dy$ (sometimes called "scaling" in numerical analysis) clearly shows the "equivalence", and hence $x^* = Dy^*$.

4. Given the following information about the initial tableau M_1 and the terminal tableau M_t for a particular application of the simplex algorithm

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$$M_1 = \begin{array}{c} \\ \left[\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ \hline 3 & -1 & 1 & 2 & 1 & 0 \\ 2 & 2 & -1 & -1 & 0 & 1 \end{array}\right] \end{array} \qquad M_t = \left[\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ 12 & 0 & 0 & 3 & 3 & 2 \\ \hline 8 & 0 & 1 & 3 & 2 & 1 \\ 5 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

$\mathcal{J} = (4,5)$ $\qquad\qquad$ $\mathcal{J} = (2,1)$

(a) determine the first pivot matrix P_1.

$$P_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) insert the missing entries in M_1, M_t and $\mathcal{J}$ (not necessarily in that order).

$\mathcal{J}$ specifies the last two columns of M_1 and hence

$Q_{t1} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, which when multiplied into the first three columns of M_1 produces the first three columns of M_t and hence both $\vec{J} = (2, 1)$ and $Q_{1t} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, which when multiplied into column 3 of M_t produces column 3 of M_1.