The Most General Linear Optimization Problems
(including unrestricted variables and/or equality constraints)

The most general linear optimization problem, conveniently defined as a linear minimization problem in terms of four coefficient matrices A_{eu}, A_{er}, A_{iu}, A_{ir}, two column vectors b^e and b^i, two coefficient row vectors c^u and c^r, a scalar d, and two column decision vector variables x^u and x^r, is as follows.

Problem P Min $f = c^u x^u + c^r x^r - d$ subject to

$$A_{eu} x^u + A_{er} x^r = b^e$$

$$A_{iu} x^u + A_{ir} x^r \leq b^i$$

$$x^u \text{ unrestricted}, \quad x^r \geq 0.$$

This linear optimization problem P includes many important special cases. For example, the constraint coefficient matrices A_{eu} and A_{iu} along with both the objective coefficient row vector c^u and the decision vector variable x^u are not present (or can all be taken to be zero) when an optimization problem has no unrestricted variables. Similarly, the constraint coefficient matrices A_{eu} and A_{er} along with the constraint column vector b^e are not present (or can all be taken to be zero) when there are no equality constraints. Of course, the most general case considered so far (the canonical case) occurs when both of the preceding cases occur. Clearly, all of the nontrivial interesting cases require the presence of at least one nonzero constraint coefficient matrix from the set of four matrices A_{eu}, A_{er}, A_{iu}, and A_{ir} -- a total of fifteen cases (some of which are much more important than the others). The following developments explicitly treat only the most general case (in which all four matrices A_{eu}, A_{er}, A_{iu}, and A_{ir} are present and are nonzero), but the reader should then be able to treat any of the fourteen other (simpler) cases.

To obtain the dual Q of this most general primal minimization problem P, first reformulate problem P as an equivalent canonical minimization problem P_C. To do so, simply replace the unrestricted vector variable x^u with the difference $x^{u+} - x^{u-}$ of two restricted vector variables $x^{u+} \geq 0$ and $x^{u-} \geq 0$; and then replace each equality constraint with two equivalent inequality constraints. The resulting canonical reformulation P_C of problem P is as follows.

<u>Problem</u> P_C Min $f = c^u x^{u+} - c^u x^{u-} + c^r x^r - d$ subject to

$$A_{eu} x^{u+} - A_{eu} x^{u-} + A_{er} x^r \leq b^e$$

$$-A_{eu} x^{u+} + A_{eu} x^{u-} - A_{er} x^r \leq -b^e$$

$$A_{iu} x^{u+} - A_{iu} x^{u-} + A_{ir} x^r \leq b^i$$

$$x^{u+}, x^{u-}, x^r \geq 0.$$

Since this minimization problem P_C is canonical, it has a dual in the sense previously defined for canonical minimization problems -- namely a minimization problem Q_C defined in terms of three row dual vector variables y^{e+}, y^{e-}, and y^i as follows.

<u>Problem</u> Q_C Min $g = y^{e+} b^e - y^{e-} b^e + y^i b^i + d$ subject to

$$-y^{e+} A_{eu} + y^{e-} A_{eu} - y^i A_{iu} \leq c^u$$

$$y^{e+} A_{eu} - y^{e-} A_{eu} + y^i A_{iu} \leq -c^u$$

$$-y^{e+} A_{er} + y^{e-} A_{er} - y^i A_{ir} \leq c^r$$

$$y^{e+}, y^{e-}, y^i \geq 0.$$

To obtain the dual Q of the original (most general) primal minimization problem P, first replace in problem Q_C the difference $y^{e+} - y^{e-}$ of the two restricted dual vector variables $y^{e+} \geq 0$ and $y^{e-} \geq 0$ by the unrestricted dual vector variable y^e; and then replace the first two sets of inequality constraints with a single set of equivalent equality constraints. The resulting reformulation Q of problem Q_C is, by definition, the dual of the original (most general) primal minimization problem P.

In summary, the <u>dual</u> of the original (most general) primal minimization problem P is as follows.

<u>Problem</u> Q Min $g = y^e b^e + y^i b^i + d$ subject to

$$-y^e A_{eu} - y^i A_{iu} = c^u$$

$$-y^e A_{er} - y^i A_{ir} \leq c^r$$

$$y^e \text{ unrestricted}, \; y^i \geq 0.$$

Note that this dual minimization problem Q can be constructed directly from the primal minimization problem P as follows (without recourse to the preceding intermediate reformulations P_C and Q_C):

(i) Interchange the roles played by the vectors $b = (b^e, b^i)$ and $c = (c^u, c^r)$,

(ii) Replace both the scalar d and the constraint coefficient matrix

$$A = \begin{bmatrix} A_{eu} & A_{er} \\ A_{iu} & A_{ir} \end{bmatrix}$$

with their negative transposes -d and

$$-A^t = \begin{bmatrix} -A_{eu}{}^t & -A_{iu}{}^t \\ -A_{er}{}^t & -A_{ir}{}^t \end{bmatrix}$$

respectively (or, equivalently, replace d with -d and multiply -A on the left by the row vector variable $y = (y^e, y^i)$, as indicated in the definition of dual minimization problem Q),

(iii) Let the dual variables y^e corresponding to the primal equality constraints be unrestricted,

(iv) Let the dual variables y^i corresponding to the primal inequality constraints be restricted,

(v) Let the dual constraints corresponding to the primal unrestricted variables x^u be equalities,

(vi) Let the dual constraints corresponding to the primal restricted variables x^r be inequalities.

Since the same construction applied to dual problem Q clearly produces primal problem P, this generalized duality correspondence is symmetric; so either problem P or problem Q could serve as the primal problem.

The <u>appropriate complementary slackness conditions</u> for the most general dual minimization problems P and Q can be obtained from the complementary slackness conditions for the equivalent intermediate canonical dual minimization problems P_C and Q_C -- via their following slack-variable formulations:

<u>Problem</u> P_C Min $f = c^u x^{u^+} - c^u x^{u^-} + c^r x^r - d$ subject to

$$A_{eu} x^{u^+} - A_{eu} x^{u^-} + A_{er} x^r + \eta^{e^+} = b^e$$

$$-A_{eu} x^{u^+} + A_{eu} x^{u^-} - A_{er} x^r + \eta^{e^-} = -b^e$$

$$A_{iu} x^{u^+} - A_{iu} x^{u^-} + A_{ir} x^r + \eta^i = b^i$$

$$x^{u^+}, x^{u^-}, x^r, \eta^{e^+}, \eta^{e^-}, \eta^i \geq 0.$$

<u>Problem</u> Q_C Min $g = y^{e^+} b^e - y^{e^-} b^e + y^i b^i + d$ subject to

$$-y^{e^+} A_{eu} + y^{e^-} A_{eu} - y^i A_{iu} + \xi^{u^+} = c^u$$

$$y^{e^+} A_{eu} - y^{e^-} A_{eu} + y^i A_{iu} + \xi^{u^-} = -c^u$$

$$-y^{e^+} A_{er} + y^{e^-} A_{er} - y^i A_{ir} + \xi^r = c^r$$

$$y^{e^+}, y^{e^-}, y^i, \xi^{u^+}, \xi^{u^-}, \xi^r \geq 0.$$

In particular, note that the complementary slackness conditions for the preceding intermediate canonical dual minimization problems P_C and Q_C are:

$$\xi^{u^+} x^{u^+} = 0,\ \xi^{u^-} x^{u^-} = 0,\ \xi^r x^r = 0,\ y^{e^+} \eta^{e^+} = 0,\ y^{e^-} \eta^{e^-} = 0,\ y^i \eta^i = 0.$$

Since $\xi^{u^+} = \xi^{u^-} \equiv 0$ for each dual feasible solution y (because $\xi^{u^+}, \xi^{u^-} \geq 0$ and because adding the first two constraint equations for the preceding formulation of problem Q_C shows that $\xi^{u^+} + \xi^{u^-} = 0$), and since $\eta^{e^+} = \eta^{e^-} \equiv 0$ for each primal feasible solution x (because $\eta^{e^+}, \eta^{e^-} \geq 0$ and because adding the first two constraint equations for the preceding formulation of problem P_C shows that $\eta^{e^+} + \eta^{e^-} = 0$), four of the preceding six displayed complementary slackness conditions are automatically satisfied for feasible solutions x and y -- and hence can be deleted. In particular then, the appropriate complementary slackness conditions for the most general dual problems P and Q are simply:

$$\xi^r x^r = 0 \text{ and } y^i \eta^i = 0.$$

In summary, there is a complementary slackness condition for each restricted variable (and its associated dual inequality constraint) and for each inequality constraint (and its associated dual restricted variable). There are no complementary slackness conditions for unrestricted variables and equality constraints.

Exercise: State the "main duality proposition" for the most general dual minimization problems P and Q. [Hint: Apply to the intermediate canonical dual minimization problems P_C and Q_C the "main duality proposition" originally established (via the simplex method) for general canonical dual minimization problems; and then interpret the results in the context of the equivalent most general dual minimization problems P and Q.]

Primal-Dual Input Schema in the Most General Case
(with unrestricted variables and/or equality constraints)

As in the case of canonical dual minimization problems P and Q, the most general dual minimization problems P and Q defined in the preceding section (embodying the most general optimization problems with unrestricted variables and/or equality constraints) can be represented by a single primal-dual input schema -- after the introduction of both slack variables and "artificial variables" that serve as the initial dependent (basic) variables.

As in the case of canonical dual minimization problems P and Q (which have only inequality constraints), slack vector variables, say $\eta^i \geq 0$ and $\xi^r \geq 0$, should be used to transform the inequality constraints

$$A_{iu} x^u + A_{ir} x^r \leq b^i \text{ and } -y^e A_{er} - y^i A_{ir} \leq c^r$$

into the equivalent equality constraints

$$A_{iu} x^u + A_{ir} x^r + \eta^i = b^i \text{ and } -y^e A_{er} - y^i A_{ir} + \xi^r = c^r$$

respectively -- with the dependent (basic) vector variables $\eta^i \geq 0$ and $\xi^r \geq 0$ respectively. However, unlike the case of canonical dual problems P and Q (which have no equality constraints prior to the introduction of the slack variables), "artificial" vector variables, say $\eta^e = 0$ and $\xi^u = 0$, should be used to transform the equality constraints

$$A_{eu} x^u + A_{er} x^r = b^e \text{ and } -y^e A_{eu} - y^i A_{iu} = c^u$$

into the equivalent equality constraints

$$A_{eu} x^u + A_{er} x^r + \eta^e = b^e \text{ and } -y^e A_{eu} - y^i A_{iu} + \xi^u = c^u$$

respectively -- with the dependent (basic) vector variables $\eta^e = 0$ and $\xi^u = 0$ respectively.

The resulting reformulations of the most general dual minimization problems P and Q defined in the preceding section (embodying the most general optimization problems with unrestricted variables and/or equality constraints) are as follows:

<u>Problem</u> P Min $f = c^u x^u + c^r x^r - d$ subject to

$$A_{eu} x^u + A_{er} x^r + \eta^e = b^e$$

$$A_{iu} x^u + A_{ir} x^r + \eta^i = b^i$$

$$x^u \text{ unrestricted, } x^r, \eta^i \geq 0, \eta^e = 0.$$

and

<u>Problem</u> Q Min $g = y^e b^e + y^i b^i + d$ subject to

$$-y^e A_{eu} - y^i A_{iu} + \xi^u = c^u$$

$$-y^e A_{er} - y^i A_{ir} + \xi^r = c^r$$

$$y^e \text{ unrestricted, } y^i, \xi^r \geq 0, \xi^u = 0.$$

These canonical reformulations of the most general dual minimization problems P and Q can be conveniently represented by the following most general primal-dual input schema

		x^u	x^r	
	$\pm d$	c^u	c^r	$-f$
y^e	b^e	$\pm A_{eu}$	$\pm A_{er}$	η^e
y^I	b^i	$\pm A_{iu}$	$\pm A_{ir}$	η^i
	$-g$	ξ^u	ξ^r	

whose primal vector variables x^u, x^r, η^e, η^i and dual vector variables y^e, y^i, ξ^u, ξ^r are classified as follows.

<u>Unrestricted</u> vector variables (components $\in$ R): x^u, y^e,

<u>Restricted</u> vector variables (components ≥ 0): x^r and ξ^r, y^i and η^i,

Artificial vector variables (components = 0): η^e, ξ^u.

The Primal and Dual Simplex Methods in the Most General Case
(with unrestricted variables and/or equality constraints)

Both simplex methods begin with a sequence of pivot operations that attempt to transform the most general primal-dual input schema (given in the preceding section) into an equivalent primal-dual schema whose primal and dual artificial variables are all independent -- so that each can be set permanently equal to zero and hence can be deleted along with its associated column or row. Since the primal and dual artificial variables (all of which are initially dependent) are paired with corresponding dual and primal unrestricted variables (all of which are initially independent), such pivot operations simultaneously attempt to transform the most general primal-dual input schema into an equivalent primal-dual schema whose primal and dual unrestricted variables are all dependent -- a desirable state because each unrestricted dependent variable can take on any value needed to make its associated equality constraint automatically satisfied, and hence can be removed along with its associated column or row.

Each pivot element in the sequence of pivot operations can, of course, be either positive or negative but should be chosen so that a dependent artificial variable is interchanged with an independent unrestricted variable or an independent restricted variable -- preferably an independent unrestricted variable because such a pivot clearly causes both a primal artificial variable and a dual artificial variable to become independent while causing both a primal unrestricted variable and a dual unrestricted variable to go from being independent to being dependent. In particular, the initial pivot element should be a non-zero element in either A_{eu}, A_{er}, or A_{iu} -- with A_{eu} preferred.

The preceding pivot-selection policies (that cause as many primal and dual artificial variables as possible to become independent while causing as many primal and dual unrestricted variables as possible to become dependent) clearly ends after a finite number of pivots -- when there can be no additional such pivots because the current primal-dual schema looks like

		unrestricted	artificial	restricted	
		= 0?			−f
artificial					unrestricted
unrestricted	= 0?	0 … 0		0 … 0	artificial
restricted		0 … 0			restricted
	−g	artificial	unrestricted	restricted	

where its columns and rows and corresponding labels have been appropriately rearranged to achieve the groupings indicated.

For primal problem P (the row problem), the columns with an artificial label can be deleted along with the corresponding artificial variables. Then, if there is a non-zero distinguished column element in a row with an artificial label, problem P must clearly be inconsistent. If not, then those rows (equality constraints) are clearly automatically satisfied and hence can be deleted. Now, those rows (equality constraints) with an unrestricted label can obviously always be satisfied for any choice of the independent variables and hence can be removed from the primal schema (but stored rather than deleted) while further considering the nature of problem P. In particular, if there is a non-zero distinguished row element in a column with an unrestricted label, problem P must clearly be unbounded if consistent. In any event, the consistency of problem P is clearly determined by the consistency of the canonical primal (row) problem whose constraint coefficient matrix is the sub-matrix in the lower right-hand corner of the displayed primal-dual schema. If the simplex method (as originally developed for canonical problems) shows that this "canonical subproblem" is consistent, then the original problem P is also clearly consistent. Moreover, if the simplex method (as originally developed for canonical problems) shows that this "canonical subproblem" is both consistent and unbounded, then the original problem P is also clearly consistent and unbounded. Furthermore, if the simplex method (as originally developed for canonical problems) shows that this "canonical subproblem" is both consistent and bounded, then the original problem P is also clearly consistent and bounded if, and only if, there is no non-zero distinguished row element in a column with an unrestricted label in the primal schema displayed above; in which case both problems clearly have the same optimal value.

Exercises: 1. Do a similar analysis for the dual problem Q (the column problem).

2. Using the preceding results concerning primal problem P along with the results obtained in Exercise 1 concerning its dual problem Q, show how the "Main Duality Proposition" originally established only for canonical problems can be extended to the most general case involving unrestricted variables and/or equality constraints.

Another Formulation of Duality

In all textbooks known to the author, the primal problem in the simplest case (with all variables being restricted and all constraints being inequalities) is formulated as follows.

Problem $\overline{P}$. Min $f \triangleq c\,x - d$ subject to

$$\overline{A}x \geq \overline{b}$$

$$x \geq 0 \ .$$

Note that problem $\overline{P}$ differs from problem P only because the constraints $\overline{A}x \geq \overline{b}$ have a reversed direction.

Since the constraints $\overline{A}x \geq \overline{b}$ are equivalent to the constraints $-\overline{A}x \leq -\overline{b}$, letting $A = -\overline{A}$ and letting $b = -\overline{b}$ makes problem $\overline{P}$ identical to our problem P. Consequently, the dual problem Q corresponding to problem $\overline{P}$ is as follows.

Min $g \triangleq -\overline{b}\,y + d$ subject to

$$\overline{A}^t y \leq c$$

$$y \geq 0$$

Instead of minimizing g, we can of course maximize $-g$. The result is, by definition, the standard dual of problem $\overline{P}$.

The <u>standard dual</u> of problem $\overline{P}$ is as follows.

<u>Problem</u> $\overline{Q}$. Max $\overline{g} \triangleq \overline{b}\,y - d$ subject to

$$\overline{A}^t y \le c$$

$$y \ge 0 \; .$$

Given problem $\overline{P}$, note that its corresponding standard dual problem $\overline{Q}$ can be directly constructed as follows:

(a) Interchange the roles played by the vectors $\overline{b}$ and c,

(b) Replace the coefficient matrix $\overline{A}$ with its transpose $\overline{A}^t$,

(c) Reverse the direction of the inequality constraints,

(d) Replace minimization by maximization.

There is, of course, symmetry in this standard formulation of duality, but it is somewhat less transparent.

Since the minimum of g is the same as minus the maximum of minus g, note that the fundamental relation $0 = f^* + g^*$ becomes

$$\text{Max } \overline{g} = \text{Min } f \; .$$

<u>Exercise</u>: 1. State and prove the "main duality proposition" for problems $\overline{P}$ and $\overline{Q}$ (using the proof for problems P and Q as a guide, if necessary).

2. Generalize the statements of problems $\overline{P}$ and $\overline{Q}$ so that unrestricted variables and equality constraints are incorporated in both $\overline{P}$ and $\overline{Q}$.

3. State the "main duality proposition" for the generalized dual problems $\overline{P}$ and $\overline{Q}$ constructed in exercise 2.