<u>Note</u>: The <u>complementary slackness</u> conditions can be written in any of the following ways:

$$\begin{cases} \text{either } \xi_j = 0 \text{ or } x_j = 0 \text{ for } j = 1,2,\ldots,m \\ \text{either } \eta_i = 0 \text{ or } y_i = 0 \text{ for } i = 1,2,\ldots,p \end{cases}$$

$$\begin{cases} \xi_j x_j = 0 \text{ for } j = 1,2,\ldots,m \\ \eta_i y_i = 0 \text{ for } i = 1,2,\ldots,p \end{cases}$$

$$\begin{cases} \sum_{j=1}^{m} \xi_j x_j = 0 \\ \sum_{i=1}^{p} \eta_i y_i = 0 \end{cases}$$

$$\left\{ \sum_{j=1}^{m} \xi_j x_j + \sum_{i=1}^{p} \eta_i y_i = 0 \right. .$$

Moreover, in each of these ways, ξ_j and η_i are frequently eliminated via the defining equations

$$\xi_j = c_j + \sum_{i=1}^{p} y_i a_{ij} \quad \text{for } j = 1,2,\ldots,m$$

$$\eta_i = b_i - \sum_{j=1}^{m} a_{ij} x_j \quad \text{for } i = 1,2,\ldots,p.$$

<u>A related complementarity problem</u>

In addition to the dual problems $\mathcal{P}$ and $\mathcal{Q}$, the main duality theorem is also concerned with the following (nonoptimization) problem.

<u>Problem</u> $\mathcal{C}$. Find all solutions (x,y) to the <u>extremality conditions (optimality conditions)</u>

$$\sum_{j=1}^{m} a_{ij} x_j \le b_i \qquad i = 1,\ldots,p$$

$$x_j \ge 0 \qquad j = 1,\ldots,m$$

$$-\sum_{i=1}^{p} y_i a_{ij} \le c_j \qquad j = 1,\ldots,m$$

$$y_i \ge 0 \qquad i = 1,\ldots,p$$

$$\text{either } c_j + \sum_{i=1}^{p} y_i a_{ij} = 0 \text{ or } x_j = 0 \text{ for } j = 1,\ldots,m$$

$$\text{either } b_i - \sum_{j=1}^{m} a_{ij} x_j = 0 \text{ or } y_i = 0 \text{ for } i = 1,\ldots,p.$$

In fact, the main duality theorem clearly implies that the following conditions are equivalent:

(i) problem P is consistent and bounded,

(ii) problem Q is consistent and bounded,

(iii) problems P and Q are consistent,

(iv) problem C has at least one solution.

The main duality theorem also implies that if any one of these conditions is satisfied (in which event all of them are satisfied), then problem C has a solution set

$$E^* = S^* \times T^*,$$

where the optimal solution sets S^* and T^* for problems P and Q respectively can, in principle, be determined via either the primal or the dual simplex method.

Efficient procedures for determining both S^* and T^* are generally based on the fact that either simplex method produces both an $x^* \varepsilon S^*$ and a $y^* \varepsilon T^*$.

Computational Considerations

If $b \ngeq 0$ and $c \ngeq 0$, then a phase 1 calculation must be used on either the primal problem P or its dual problem Q. Computationally, it is frequently best to use a phase 1 calculation on the problem with the smallest number of constraints (rather than on the problem with the smallest number of variables), because pivoting consists essentially of inverting certain matrices whose size is one larger than the number of constraints.

If the subsequent phase 2 calculation terminates with step 1, the terminal schema (or tableau) produces both an $x^* \in S^*$ and a $y^* \in T^*$. To identify such an x^* and y^*, simply recall that the labels x_j and ξ_j must be opposite one another in the terminal schema, as must the labels y_i and η_i.

Needless to say, using the primal simplex algorithm to pivot in those columns whose distinguished row element is zero can produce other primal optimal solutions. Likewise, using the dual simplex algorithm to pivot in those rows whose distinguished column element is zero can produce other dual optimal solutions.

Finally, it is worth noting that phase 2 of the simplex algorithm (either primal or dual) maintains complementary slackness along with feasibility for one problem while seeking feasibility for the other problem.

Determination of a Dual Optimal Solution Directly from a Terminally Optimal Tableau

Since a terminally optimal tableau M_t can be expressed in schema form, resulting in a terminally optimal schema that produces both a primal optimal solution x^* and a dual optimal solution y^*, it is clear that y^* should be as easily obtainable directly from M_t as is x^*.

To see how to obtain y^* directly from M_t, first note that the input tableau for problem P is

$$M_0 = \begin{array}{c} \begin{array}{ccc} & x & \eta \end{array} \\ \left[\begin{array}{c|cc} d & c & 0 \\ \hline b & A & I \end{array}\right] \end{array} \qquad \sigma = (m+1, m+2, \ldots, m+p).$$

Then, note that the special nature of this input basic sequence σ implies that the terminal tableau has the special form

$$M_t = \begin{array}{c} \begin{array}{ccc} & x & \eta \end{array} \\ \left[\begin{array}{c|cc} d_t & c_t^{\sigma'} & c_t^{\sigma} \\ \hline b_t & A_t^{\sigma'} & A_t^{\sigma} \end{array}\right] \end{array} T .$$

Now, note that this terminal tableau M_t with basic sequence T can be expressed in schema form as

$$\begin{array}{|c|c|} \hline d_t & c_t^{T'} \\ \hline b_t & A_t^{T'} \\ \hline \end{array} \; ,$$

with the missing labels to be inserted in accordance with T.

Using the preceding terminal tableau and schema, we can show that *the dual optimal solution y^* produced by the terminal schema is simply the vector $c_t^{T'}$ appearing in the upper right-hand corner of the corresponding terminal tableau.* To do so, we need to consider two mutually exclusive and exhaustive cases having to do with a given component y_i^* of y^*. In particular, if y_i is terminally non-basic, then y_i^* is zero; and since the relative position of y_i and η_i in the input primal-dual schema for problems P and Q remains invariant in the terminal primal-dual schema, η_i is terminally basic, which implies that the $i^{\underline{th}}$ component of $c_t^{T'}$ is also zero by virtue of its position under the label η_i in the terminal tableau M_t. On the other hand, if y_i is terminally basic, the relative position invariance of y_i and η_i implies that y_i^* is the component of $c_t^{T'}$ appearing under the label η_i in the terminal schema; and since that component of c_t also appears under the label η_i in the terminal tableau M_t, it is simply the $i^{\underline{th}}$ component of $c_t^{T'}$. In summary then, we have shown that $y^* = c_t^{T'}$ -- an equation that is fundamental in relating the dual optimal solution y^* to a post-optimal sensitivity analysis of primal problem P.

Post-optimal Sensitivity Analysis Revisited

The general formulas derived in the preceding chapter for $\nabla_b f^*$ and $\nabla_c f^*$ can now be given useful interpretations in the special context of dual problems P and Q. Such interpretations can be obtained via the fundamental relation $y^* = c_t^{T'}$ derived in the preceding section.

In particular, note that statement (I') in the preceding chapter can now be rephrased in this special context as

$$\nabla_b f^* = -y^* \text{ \underline{when} } x^* \text{ \underline{is} \underline{not} \underline{degenerate}},$$

where b has been replaced by its equivalent symbol b. Dually, $\nabla_c g^* = -x^*$ when y^* is not degenerate; and since $0 \equiv f^* + g^*$, we know that $\nabla_c f^* + \nabla_c g^* \equiv 0$. Consequently,

$$\nabla_c f^* = x^* \text{ \underline{when} } y^* \text{ \underline{is} \underline{not} \underline{degenerate}},$$

which is simply statement (II') in the preceding chapter rephrased in this special duality context.

The preceding displayed statements will be generalized in a later chapter to the unconditional statements $\partial_b f^* = -T^*$ and $\partial_c f^* = S^*$, which involve generalized gradients termed "subgradients."

Optimal Solution Set Characterizations

If the simplex method (either primal or dual) terminates in phase 2 with step 1, we know that the terminal schema (or tableau) produces both an $x^* \epsilon S^*$ and a $y^* \epsilon T^*$. Moreover, we also know that if y^* is not degenerate, then $S^* = \{x^*\}$; and, dually, if x^* is not degenerate, then $T^* = \{y^*\}$. In all other cases, a single x^* and a single y^* is still all that is needed to provide useful formulas for S^* and T^*, because the main duality proposition clearly implies that

$$S^* = \{x \mid (x, y^*) \epsilon E^*\} \text{ for any given } y^* \epsilon T^* .$$

and

$$T^* = \{y \mid (x^*, y) \epsilon E^*\} \text{ for any given } x^* \epsilon S^* .$$

The preceding formula for S^* does not require as input a complete knowledge of y^*, but only a knowledge of the corresponding dual-active constraints expressed in terms of the "dual-active index sets"

$$I_Q \triangleq \{i \mid y_i^* = 0\} \text{ and } J_Q \triangleq \{j \mid -\sum_{i=1}^{p} y_i^* a_{ij} = c_j\} .$$

In particular, the preceding formula simply asserts that S^* is the set of all solutions x to the following restriction of the primal constraints

$$\sum_{J_Q} a_{ij}\, x_j \leq b_i \qquad i \in I_Q$$

$$\sum_{J_Q} a_{ij}\, x_j = b_i \qquad i \notin I_Q$$

$$x_j \geq 0 \qquad j \in J_Q$$

$$x_j = 0 \qquad j \notin J_Q \;.$$

Dually, the preceding formula for T* does not require as input a complete knowledge of x^*, but only a knowledge of the corresponding primal-active constraints expressed in terms of the "primal-active index sets"

$$I_P \triangleq \{i \mid \sum_{j=1}^{m} a_{ij}\, x_j^* = b_i\} \text{ and } J_P \triangleq \{j \mid x_j^* = 0\}.$$

In particular, the preceding formula simply asserts that T^* <u>is the set of all solutions</u> y <u>to the following restriction of the dual constraints</u>

$$-\sum_{I_P} y_i a_{ij} \leq c_j \qquad j \in J_P$$

$$-\sum_{I_P} y_i a_{ij} = c_j \qquad j \notin J_P$$

$$y_i \geq 0 \qquad i \in I_P$$

$$y_i = 0 \qquad i \notin I_P \;.$$

Note that the preceding characterizations for S* and T* use, in essence, the primal and dual systems of equations represented by the input schema (with the slack variables omitted and hence with the inactive constraints written as inequalities rather than equations). Since those primal and dual systems of equations are equivalent, respectively, to the primal and dual systems of equations represented by the terminal schema for phase 2, S* and T* can also be characterized in terms of that terminal schema.

To derive such a characterization for S*, first note from the defining formulas for I_Q and J_Q that both $y_i^* > 0$ for $i \notin I_Q$ and $\xi_j^* > 0$ for $j \notin J_Q$, which implies that both y_i for $i \notin I_Q$ and ξ_j for $j \notin J_Q$ are terminal basic variables (because the terminal nonbasic variables are zero). Hence, both η_i for $i \notin I_Q$ and x_j for $j \notin J_Q$ are terminal nonbasic variables (by virtue of the relative position invariance of primal and dual labels). Moreover, for primal optimal solutions x, both $\eta_i \equiv 0$ for $i \notin I_Q$ and $x_j \equiv 0$ for $j \notin J_Q$ (by virtue of the complementary slackness conditions and the fact that both $y_i^* > 0$ for $i \notin I_Q$ and $\xi_j^* > 0$ for $j \notin J_Q$). Consequently, both η_i for $i \notin I_Q$ and x_j for $j \notin J_Q$, as well as their corresponding columns, can be deleted from the terminal primal schema -- resulting in a deleted primal schema whose remaining labels are η_i for $i \in I_Q$ and x_j for $j \in J_Q$, as well as -f. Then, the non-negativity constraints $\eta_i \geq 0$ for $i \in I_Q$ and $x_j \geq 0$ for $j \in J_Q$, along with the system of equations represented by the deleted primal schema, characterize the primal optimal solutions x* (actually, their non-identically zero components). Moreover, since the deleted primal schema clearly has either no distinguished row or only zero entries in its distinguished row, its distinguished element and any distinguished row elements can be deleted along with the row label -f -- leaving a deleted primal constraint schema that can be used in place of the deleted primal schema.

In summary, to characterize S* in terms of the terminal primal schema for phase 2:

1. Delete each column that is headed by a nonzero distinguished row element, and set each of the corresponding primal labels permanently equal to zero.
2. Delete the resulting row zero and its corresponding primal label -f.

Then, the row equations of the resulting deleted primal constraint schema along with the non-negativity constraints for the undeleted primal labels characterize the primal optimal solutions x*, and hence S*. In fact, S* contains a single element x* if this schema has a single column (i.e., has no nonbasic primal variables). Otherwise, each nondegenerate pivot using steps 2 and 3 of the primal simplex algorithm on this schema produces an additional basic optimal solution x* in S*.

Dually, to characterize T* in terms of the terminal dual schema for phase 2:

1. Delete each row that is headed by a nonzero distinguished column element, and set each of the corresponding dual labels permanently equal to zero.
2. Delete the resulting column zero and its corresponding dual label -g.

Then, the column equations of the resulting deleted dual constraint schema along with the non-negativity constraints for the undeleted dual labels characterize the dual optimal solutions y^*, and hence T^*. In fact, T^* contains a single element y^* if this schema has a single row (i.e., has no nonbasic dual variables). Otherwise, each nondegenerate pivot using steps 2 and 3 of the dual simplex algorithm on this schema produces an additional basic optimal solution y^* in T^*.

The preceding formulas for S^* and T^*, like the defining formulas for S^* and T^*, involve only linear inequalities and equations. However, since the preceding formulas are obviously simpler than the defining formulas, the preceding formulas will be used in conjunction with techniques based on the more thoroughly developed theory in later chapters.

Finally, it is worth noting that there is a way of simultaneously characterizing all primal optimal solutions x^* and all dual optimal solutions y^* via linear inequalities that do not require as input any information from a terminal schema (in particular, do not require either a knowledge of the primal and dual optimal values f^* and g^* or a knowledge of primal and dual active index sets I_P, J_P and I_Q, J_Q). In particular, since the strong duality theorem asserts for a consistent bounded problem P (or Q) that $0 = f^* + g^*$, the weak duality theorem implies that $x \in S^*$ and $y \in T^*$ if and only if

$$\sum_{j=1}^{m} a_{ij}\, x_j \leq b_i \qquad i = 1, \ldots, p$$

$$x_j \geq 0 \qquad j = 1, \ldots, m$$

$$-\sum_{i=1}^{p} y_i a_{ij} \leq c_j \qquad j = 1, \ldots, m$$

$$y_i \geq 0 \qquad i = 1, \ldots, p$$

$$(c\, x - d) + (y\, b + d) \leq 0 .$$

This system of linear inequalities can be solved via the "ellipsoid algorithm" -- a polynomially bounded alternative to the simplex algorithm that was discovered by Khachian in the early 1980's.