

The case of an "infeasible input schema"  
 (in which  $x_1 = x_2 = \dots = x_m = 0$  is infeasible,  
 because at least one  $b_i < 0$ )

Problems with an infeasible input schema (e.g., the diet problem) might not have a feasible solution; so the question of consistency (i.e., whether feasible solutions exist) must be answered before the question of optimality (i.e., whether the problem has feasible solutions that are optimal or unbounded).

We shall now see that the question of consistency can be answered by a relatively simple modification of the simplex algorithm (previously developed for answering the question of optimality). This simple modification is called "phase 1" of the simplex method, while the simplex algorithm itself is called "phase 2" of the simplex method. As with our previous description of phase 2, we initially describe phase 1 in contracted form (using schemas).

For purposes of illustration, we use as an example the diet problem with input schema

	$x_1$	$x_2$	$x_3$	
0	2	2	10	-C(min)
-13	-2	-4	-2	$x_4$
-10	-2	-1	-3	$x_5$
-2	0	0	-1	$x_6$
5/2	1	0	0	$x_7$

The corresponding basic solution (in which  $x_1 = x_2 = x_3 = 0$ ) is infeasible, because the distinguished column has three negative elements. Consequently, phase 2 is not yet meaningful, and will never be meaningful if phase 1 shows that there are no feasible solutions (and hence no basic feasible solutions or corresponding feasible schemas representing the problem).

Phase 1 consists of a finite number of repetitions of the following four steps, which can be viewed as attempts via pivoting to increase the negative elements in the distinguished column to nonnegative values while preventing any currently nonnegative elements from becoming negative. (Parenthetical statements explain the validity of steps 1 and 2; while steps 3 and 4 are algebraically identical to steps 3 and 4 respectively of the simplex algorithm, and hence are valid by virtue of elementary linear algebra.)

Step 1. Choose any negative element in the distinguished column. Then, in the row of that negative element, choose any other negative element. (If no such other negative element exists, the corresponding constraint can clearly never be satisfied for  $x \geq 0$ ; so the problem is inconsistent.)

Step 2. Suppose step 1 gives negative elements  $b_t$  and  $a_{tk}$  in the  $t$ 'th row and  $0$ 'th and  $k$ 'th columns respectively. Then, clearly the following two scenarios, and only the following two scenarios, can occur:

Scenario 1 [in which all positive elements  $a_{ik}$  in the  $k$ 'th column, if any, correspond to negative elements  $b_i$  in the distinguished column]. Form all ratios  $b_i/a_{ik}$  for which both  $b_i < 0$  and  $a_{ik} < 0$  (including the ratio  $b_t/a_{tk} > 0$ ). That element, say  $a_{hk}$ , which produces the largest positive ratio  $b_h/a_{hk}$  is the pivot element. (If the largest ratio is not unique, any element  $a_{hk}$  that produces the largest ratio  $b_h/a_{hk}$  can be used as the pivot element.)

Scenario 2 [in which there is at least one positive element  $a_{ik}$  in the  $k$ 'th column corresponding to a nonnegative element  $b_i$  in the distinguished column]. Form all ratios  $b_i/a_{ik}$  for which both  $b_i \geq 0$  and  $a_{ik} > 0$ . That element, say  $a_{hk}$ , which produces the smallest nonnegative ratio  $b_h/a_{hk}$  is the pivot element. (If the smallest ratio is not unique, any element  $a_{hk}$  that produces the smallest ratio  $b_h/a_{hk}$  can be used as the pivot element; but the prevention of circling can not be guaranteed unless lexicographic minimization or some other anti-circling procedure is used to break the ties.)

Step 3. Same as for the simplex algorithm

Step 4. Same as for the simplex algorithm

Phase 1, like phase 2, need not produce a unique sequence of schemas from a given input schema. In particular, each of the following two sequences of schemas comes from choosing the negative element -13 in row 1 of the distinguished column of the input schema for the given diet problem. However, the first sequence of schemas comes from choosing in that row 1 the first negative element -2 while the second sequence of schemas comes from choosing in that row 1 the negative element -4. The reader should check the validity and accuracy of the resulting two solution sequences for phase 1, as follows.

		↓ $x_1$	$x_2$	$x_3$			$x_1$	↓ $x_2$	$x_3$		
→	0	2	2	10	-C(min)	→	0	2	2	10	-C(min)
	-13	-2	-4	-2	$x_4$		-13	-2	-4	-2	$x_4$
	-10	-2	-1	-3	$x_5$		-10	-2	-1	-3	$x_5$
	-2	0	0	-1	$x_6$		-2	0	0	-1	$x_6$
	5/2	1	0	0	$x_7$		5/2	1	0	0	$x_7$

	$x_7$	$x_2$	$x_3$	
-5	-2	2	10	-C(min)
-8	2	-4	-2	$x_4$
-5	2	-1	-3	$x_5$
-2	0	0	-1	$x_6$
5/2	1	0	0	$x_1$

	$x_1$	$x_5$	$x_3$	
-20	-2	-2	4	-C(min)
27	6	4	10	$x_4$
10	2	-1	3	$x_2$
-2	0	0	-1	$x_6$
5/2	1	0	0	$x_7$

	$x_7$	$x_2$	$x_4$	
-45	8	-18	5	-C(min)
4	-1	2	-1/2	$x_3$
7	-1	5	-3/2	$x_5$
2	1	2	-1/2	$x_6$
5/2	1	0	0	$x_1$

	$x_1$	$x_5$	$x_4$	
-30.8	-4.4	-3.6	-4	-C(min)
2.7	.6	.4	.1	$x_3$
1.9	.2	-2.2	-3	$x_2$
.7	.6	.4	.1	$x_6$
2.5	1	0	0	$x_7$

End of phase 1

End of phase 1

Note that for both schema sequences phase 1 has ended with a feasible input schema for phase 2. We suggest that the reader apply phase 2 (the simplex algorithm) to each of these two schemas to answer the (now relevant) question of optimality. We also suggest that the reader repeat phase 1 by still choosing element -13 in row 1 of the distinguished column of the infeasible input schema, but now choosing in that row 1 the second negative element -2, in order to see that the required number of iterations in phase 1 depends not only on the problem, but also on the pivot selection.

Exercises: 1. Prove that each nonnegative  $b_i$  is replaced by a nonnegative  $b_i'$  after step 3. (Hint: Consider each scenario separately, and pattern your proofs after the analogous proof given in the text for the simplex algorithm.)

2. Which of the two scenarios in step 2 tends to produce the most progress toward a feasible schema and why? (Hint: Which scenario guarantees that at least one negative  $b_i$  is replaced with a positive  $b_i'$  after the resulting pivot step 3?)

3. Can you describe a type of infeasible schema arising in phase 1 that requires only one iteration to produce a feasible input schema for phase 2? (Hint: Suppose that scenario 1 occurs in such a way that  $a_{ik} < 0$  for each  $i$  for which  $b_i < 0$  -- a situation that occurs, for example, when the previously suggested repetition of phase 1 for the diet-problem example is performed.)

4. Why does scenario 1 in step 2 not need an anti-circling procedure to break the ties that could occur in determining the pivot row? (Hint: Think about the consequences of exercise 2. In particular, think about the extreme case alluded to in exercise 3.)

5. If lexicographic minimization is used to break the ties that could occur in determining the pivot row during scenario 2 in step 2 (and hence prevent circling):

- Which rows need to be lexicographically positive in the input tableau?
- Which row always changes in the same lexicographic direction, and what is that direction?
- Does row 0 always change in the same lexicographic direction (as it does in the simplex algorithm when lexicographic minimization is used to break the ties that could occur in determining the pivot row in step 2)?

### Termination of Phase 1

Since there are only a finite number of possible schemas, if circling does not occur (which is guaranteed to be the case if an anticircling procedure is used in scenario 2), phase 1 must terminate after a finite number of repetitions of steps 1 through 4. In particular, phase 1 terminates at either step 1 or with a feasible schema.

If phase 1 terminates at step 1, no feasible solution exists and hence the problem is inconsistent; so phase 2 is irrelevant.

If phase 1 terminates with a feasible schema, a basic feasible solution is obtained (by setting the terminal nonbasic variables to zero), and the feasible schema can serve as the input schema for phase 2.

### Observations and Comments

Note that phase 1 can be performed without an objective function -- simply by omitting row 0 in the schemas (or tableaus). We shall eventually see that such a phase 1 can be used to determine the consistency of any finite system of linear equations and/or linear inequalities -- an important extension of the Gaussian elimination or row reduction that is the key to the study of "linear algebra" (which deals only with linear equations).

If phase 1 is performed in uncontracted form (using tableaus), pivot matrices  $P_q$  and their products  $Q_{sr}$  can be used in a "revised version" to reduce data-storage requirements and computational effort.

Other versions of phase 1 (including the original version using "artificial variables" that is probably still used in many, if not all, commercial computer software packages) can also be used, but they are not as easily motivated, described, or validated as the version given here. A combination of phase 1 and phase 2 using artificial variables, called the "big M method", actually goes through the same pivots as a phase 1 followed by a phase 2 and hence has no computational advantage over other versions of the simplex method. In fact, certain scenarios can occur that, to the best of the author's knowledge, prevent the construction of a proof of termination in a finite number of iterations; so the big M method actually seems to be inferior to the other methods.