SIMPLEX ALGORITHM IN CONTRACTED FORM

We begin by treating problems that have already been placed in canonical form. Eventually, we will see how to treat all other linear optimization problems.

For notational convenience, we assume that the basic variables are enumerated last. In particular, we consider the following maximization problem (with minimization problems to be considered later).

$$\text{Max } f = \sum_{j=1}^{m} c_j x_j - d \qquad \text{subject to}$$

$$\sum_{j=1}^{m} a_{ij} x_j + x_{m+i} = b_i \qquad i = 1, \ldots, p$$

$$x_j \geq 0 \qquad j = 1, \ldots, m+p$$

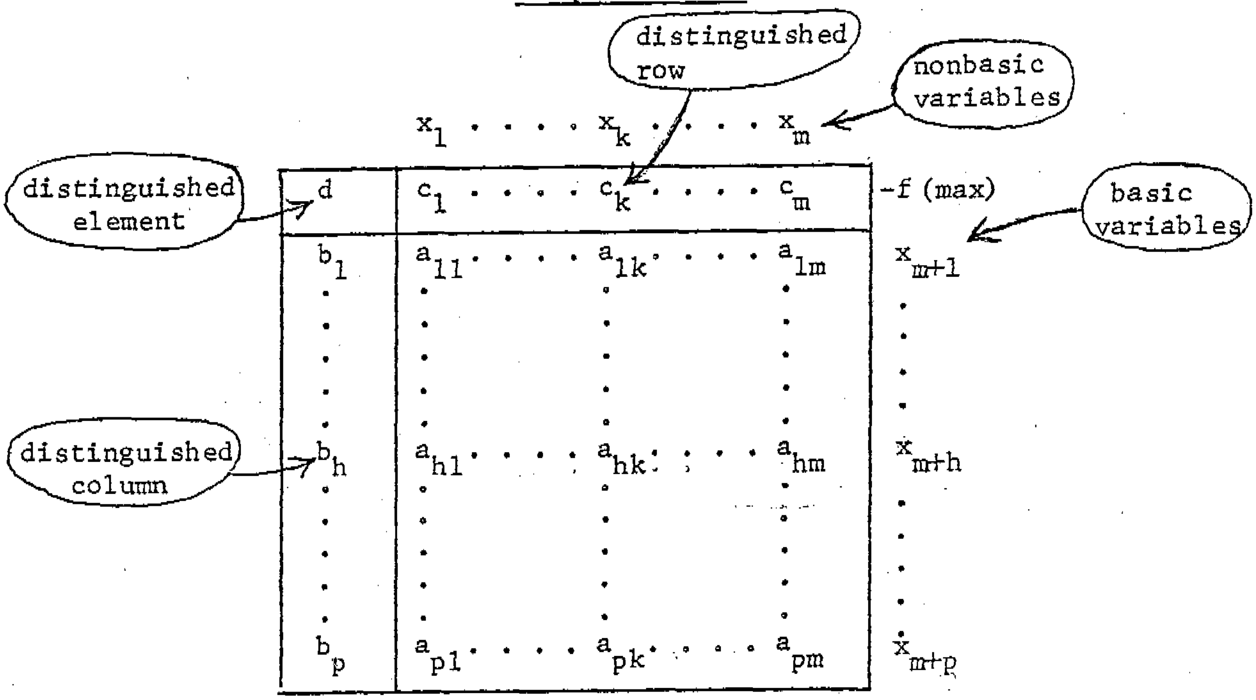
Example: Max $f = -x_1 + 2x_2 - 2x_3 - 3$ subject to

$$\begin{aligned}
-x_1 + x_2 - x_3 + x_4 &= -2 \\
2x_1 + x_2 - x_3 + x_5 &= -1 \\
-2x_1 - x_2 + x_3 + x_6 &= 1 \\
-2x_1 + 3x_2 - 3x_3 + x_7 &= 1
\end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,\ x_7 \geq 0$$

This problem can be conveniently represented in the following way.

Input schema



Example:

	x_1	x_2	x_3	
3	-1	2	-2	-f(max)
-2	-1	1	-1	x_4
-1	2	1	-1	x_5
1	-2	-1	1	x_6
1	-2	3	-3	x_7

Needless to say, each row of an input schema is to be interpreted as an equality (i.e., an equation) from the problem statement. Of course, the inequalities from the problem statement are not explicitly incorporated into an input schema because they are all of simple form $x_j \geq 0$.

The case of a "feasible input schema"

(in which $x_1 = x_2 = \ldots = x_m = 0$ is

feasible, because each $b_i \geq 0$)

Problems with a feasible input schema (i.e., problems for which each $b_i \geq 0$) can be solved by a version of the simplex algorithm that consists of a finite number of repetitions of the following four steps. (Parenthetical statements help explain the validity of steps 1 and 2, but a verification of the validity of steps 3 and 4, and hence the validity of the algorithm, must be postponed until later in this chapter.)

Step 1. Choose any positive element in the distinguished row. (If no such positive element exists, the maximum value for the linear optimization problem is clearly $-d$, and an optimal solution is $x_j = 0$, $j = 1,2, \ldots, m$ while $x_{m+i} = b_i$, $i = 1,2, \ldots, p$.)

Step 2. Suppose step 1 gives the element c_k in the k'th column. Form all ratios b_i/a_{ik} for which $a_{ik} > 0$. (If all elements a_{ik} of the k'th column are nonpositive, it is clear that f is not bounded away from $+\infty$ because x_k can be made arbitrarily large without violating the constraints; in which case the linear optimization problem has no maximum value and is 'unbounded'.) That element, say a_{hk}, which produces the smallest (non-negative) ratio b_h/a_{hk} is termed the "pivot element". (If the smallest ratio is not unique, any element a_{hk} that produces the smallest ratio b_h/a_{hk} can be used as the pivot element.)

Step 3. (a) Interchange the roles of x_k and x_{m+h}; that is, relabel the row and column of the pivot element by interchanging the symbols x_k and x_{m+h} (while letting the position of all other symbols unchanged).

(b) Replace the pivot element by its reciprocal.

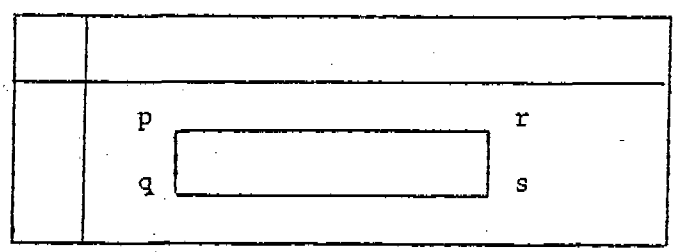
(c) Replace the other elements in the row of the pivot element by the quotient (the row element/pivot element).

(d) Replace the other elements in the column of the pivot element by the quotient (minus the column element/pivot element).

(e) Replace all other elements s by elements of the form

$$s' = \frac{ps - qr}{p} = s - \left(\frac{q}{p}\right) r$$

where p is the pivot element and q and r are schema elements for which prsq forms a rectangle as shown below.



Step 4. Use the new schema created in step 3 and return to step 1.

Example: Max $f = x_1 + x_2 + x_3 + 1$ subject to

$$\begin{aligned} x_1 - x_2 + 2x_3 + x_4 &= 2 \\ -x_1 + 2x_2 - x_3 + x_5 &= 1 \\ 2x_1 + x_2 + 2x_3 + x_6 &= 13 \end{aligned}$$

$x_j \geq 0 \quad j = 1,2,3,4,5,6$

Input schema

	x_1	x_2	x_3	
-1	1	1	1	$-f$
2	1	-1	2	x_4
1	-1	(2)	-1	x_5
13	2	1	2	x_6

New schema
(after one iteration)

	x_1	x_5	x_3	
-3/2	3/2	-1/2	3/2	$-f$
5/2	(1/2)	1/2	3/2	x_4
1/2	-1/2	1/2	-1/2	x_2
25/2	5/2	-1/2	5/2	x_6

Terminal schema
(after two iterations)

	x_4	x_5	x_3	
-9	-3	-2	-3	$-f$
5	2	1	3	x_1
3	1	1	1	x_2
0	-5	-3	-5	x_6

Optimal value $f^* = 9$ and optimal solution $x^* = (5,3,0,0,0,0)$, or $x^* = (5,3,0)$ if x_4, x_5 and x_6 happen to be slack variables.

Note: Although the formula $s' = (p\,s - q\,r)/p$ is especially easy to remember (because of the determinant-like nature of its numerator), the more concise formula $s' = s - (q/p)r$ is generally more efficient for actually performing part (e) of step 3. In addition to requiring fewer arithmetical calculations, the latter formula uses the expression (q/p), which clearly remains invariant during a computation of all elements s' in a given row (resulting essentially in an "elementary row operation" on the matrix that comes from ignoring the column of the pivot element). Other formulas for s' will also be used--to verify certain implications of "pivoting" (i.e., performing step 3).

Exercise: 1. Solve the preceding example by initially pivoting in the column of x_1, and then solve it once again by initially pivoting in the column of x_3. Can you reach any conclusions about whether the required number of iterations depends only on the problem?

2. Derive a formula for s' that shows why part (e) of step 3 can also be viewed as certain "elementary column operations" on the matrix that comes from ignoring the row of the pivot element.

Justification of step 3

Solution of the h'th constraint equation for x_k gives

$$\left(\sum_{\substack{j=1\\ j\neq k}}^{m} \frac{a_{hj}}{a_{hk}} x_j + \frac{1}{a_{hk}} x_{m+h}\right) + x_k = \frac{b_h}{a_{hk}}.$$

Substitution of the resulting expression for x_k into the other constraint equations and the objective function gives

$$\left(\sum_{\substack{j=1\\ j\neq k}}^{m} \frac{a_{hk}a_{ij} - a_{ik}a_{hj}}{a_{hk}} x_j - \frac{a_{ik}}{a_{hk}} x_{m+h}\right) + x_{m+i}$$

$$= \frac{a_{hk}b_i - a_{ik}b_h}{a_{hk}} \qquad i = 1,2,\ldots,p \qquad i \neq h$$

and

$$f = \sum_{\substack{j=1\\ j\neq k}}^{m} \frac{a_{hk}c_j - c_k a_{hj}}{a_{hk}} x_j - \frac{c_k}{a_{hk}} x_{m+h} + \frac{-a_{hk}d + c_k b_h}{a_{hk}}$$

Now, treat x_k as a basic variable while treating x_{m+h} as a nonbasic variable. The resulting schema is identical to the new schema constructed in step 3.

Observations and Additional Terminology

The new schema represents a system of equations that is equivalent to the system of equations represented by the input schema (i.e., both systems have the same solution set). Consequently, the two schemas represent equivalent problems (problems that are, in fact, identical).

Since the input schema has distinguished column elements $b_i \geq 0$ and since the intital pivot element $a_{hk} > 0$, the new schema also has distinguished column elements $b_i' \geq 0$, because

$$b_h' = \frac{b_h}{a_{hk}} \geq 0,$$

$$b_i' = \frac{a_{hk}\, b_i - a_{ik}\, b_h}{a_{hk}} \geq 0 \quad \text{for } i \neq h \text{ and } a_{ik} \leq 0,$$

and

$$b_i' = \frac{a_{hk}\, b_i - a_{ik}\, b_h}{a_{hk}} = a_{ik} \left(\frac{b_i}{a_{ik}} - \frac{b_h}{a_{hk}} \right) \geq 0 \quad \text{for } i \neq h \text{ and } a_{ik} > 0,$$

with the latter inequality resulting from the choice of "pivot row" h (i.e., step 2). Consequently, the new schema is also "feasible", in that the corresponding "basic solution" (obtained by setting each new nonbasic variable to zero) is a feasible solution, termed a "basic feasible solution" or feasible "extreme point" (and represented by the new schema).

The value of the objective function at this basic feasible solution is, of course,

$$-d' = - \frac{a_{hk}d - c_k\, b_h}{a_{hk}} = -d + \left(\frac{c_k}{a_{hk}} \right) b_h \geq -d,$$

with the latter inequality resulting from the choice of "pivot column" k (i.e., step 1). In particular then, the objective function has not decreased in value and has in fact increased in value if $b_h > 0$.

Note though that if there is an index i such that $b_i = 0$ and $a_{ik} > 0$ then $b_h = 0$ by virtue of the choice of pivot row h. Moreover, if $b_h = 0$, the displayed formulas in the preceding paragraph show that $b' = b$ and

$-d' = -d$, with the new schema representing the same basic feasible solution as its predecessor. During such a "degenerate iteration" of the simplex algorithm there is no apparent progress toward optimality. However, the new schema has the distinguished column element

$$c_k' = \frac{-c_k}{a_{hk}} < 0$$

by virtue of the choice of pivot column k; so the new pivot element (if any) can not be in column k, and hence the next schema can not be identical to the preceding schema. Yet, the next schema may still represent the same basic feasible solution. In fact, each succeeding schema may represent the same basic feasible solution; in which case "circling" back to a previous schema (sometimes called "cycling") must occur, because there are only a finite number of possible schemas (clearly no more than the number of permutations of m + p variables taken m at a time, multiplied by the number of permutations of m + p variables taken p at a time).

Circling is, of course, undesirable because it prevents one from reaching optimality. Note though that it can occur only if there are degenerate iterations (otherwise, the objective function increases with each iteration and hence can not circle back to a previous value). Since we have already seen that a degenerate iteration can occur only if the schema at the start of such an iteration has at least one zero appearing as a distinguished column element, circling can occur only if there is at least one such "degenerate feasible schema" (with the corresponding basic feasible solution being termed a "degenerate basic feasible solution"). Such a degenerate feasible schema need not appear initially, but could in fact appear after pivoting with a nondegenerate feasible schema (as indicated by Exercise 1 at the end of this section).

Although circling can occur only if there are degenerate feasible schemas, the existence of such a schema does not guarantee circling (as indicated by Exercise 2 at the end of this section). In fact, to the best of the author's knowledge, circling has never been encountered in a practical problem (though degenerate iterations followed eventually by a non-degenerate iteration are frequently encountered). In any event, since the possibility of circling has been demonstrated with a carefully constructed example, we shall eventually learn how to refine the simplex algorithm so that circling is no longer possible (even though degenerate iterations followed eventually by a nondegenerate iteration are still possible).

Exercise: 1. Given a linear optimization problem with a feasible input schema that is nondegenerate, under what conditions would one iteration of the simplex algorithm produce a degenerate schema? (Hint: Examine the previously given formulas for b_h' and b_i' to determine the conditions under which either $b_h' = 0$ or some other $b_i' = 0$, given that $b_h > 0$ and each other $b_i > 0$.)

2. Given a linear optimization problem with a feasible input schema that is degenerate, under what conditions would one iteration of the simplex algorithm produce a nondegenerate schema? (Hint: Examine Step 2 of the simplex algorithm and then the previously given formulas for b_h' and b_i'.)

Termination of the algorithm

Since there are only a finite number of possible schemas, if circling does not occur, the algorithm must terminate after a finite number of repetitions of steps 1 through 4; in which event the algorithm terminates at either step 1 or step 2.

If the algorithm terminates at step 1, at least one optimal solution is at hand. That optimal solution is clearly unique if all elements in the distinguished row of the terminal schema are negative. However, if at least one of those elements is zero, pivoting in the corresponding columns may produce additional optimal solutions (possibly even infinite rays of optimal solutions). In any event, the set of all optimal solutions also clearly contains all "convex combinations" of such optimal solutions, that is, any linear combination $\alpha_1 x^1 + \alpha_2 x^2 + \ldots + \alpha_q x^q$ of optimal solutions x^k for which $\alpha_1 \geq 0$, $\alpha_2 \geq 0, \ldots, \alpha_q \geq 0$ and $\alpha_1 + \alpha_2 + \ldots + \alpha_q = 1$(a fact that we shall elaborate on in a later chapter).

If the algorithm terminates at step 2, feasible solutions that make the objective function arbitrarily large are obtained, and hence there is no optimal solution; that is, the problem is "unbounded".

Exercise: 1. (a) Solve the linear optimization problem represented by the following feasible schema

		x_1	x_4	
	−2	1	−2	−f(max)
	0	1	−1	x_3
	3	8	1	x_2

(b) Does Part (a) shed any light on the validity of the converse of the parenthetical statement in Step 1 of the simplex algorithm; that is, if the (negative of the) distinguished element for a given feasible schema gives the maximum value for the corresponding linear optimization problem, do the distinguished row elements of the given feasible schema have to be nonpositive?

2. Consider the feasible schema

	x_1	x_4	
-5	2	1	-f(max)
3	-1	1	x_3
0	2	0	x_5
1	-2	1	x_2

(a) Does this feasible schema itself (without any pivoting) give unusual information about the nature of the corresponding feasible solution set (and hence the corresponding optimal solutions)?
[Hint: The equation causing degeneracy has no negative coefficients, which implies something about its non-negative solutions.]

(b) Using the results of Part (a), justify a replacement of the given feasible schema with the feasible schema

	x_4	
-5	1	-f(max)
3	1	x_3
0	0	x_5
1	1	x_2

Then, justify a replacement of this schema with the feasible schema

	x_4	
-5	1	$-f$(max)
3	1	x_3
1	1	x_2

(c) Solve the problem represented by the preceding feasible schema (at the end of Part (b)), and then use its optimal solution to deduce the optimal solutions to the problem represented by the feasible schema first given in this exercise.

(d) Solve, in the usual way, the problem represented by the feasible schema first given in this exercise.

(e) What are the relative advantages and disadvantages of the solution methods employed in Parts (c) and (d)?

3. Using your experience from Exercise 2, state a theorem concerning the feasible solution set (and hence any optimal solutions) for a problem represented by a degenerate feasible schema in which a row causing the degeneracy has no negative entries but at least one positive entry.

4. Can you describe a schema that represents a problem for which one can immediately infer that no feasible solutions (and hence no optimal solutions) exist? [Hint: Such a schema must necessarily have at least one negative distinguished column element. Why? Under what conditions would the constraint equation with such a negative distinguished column element have no non-negative solutions?]