LINEAR OPTIMIZATION MODELS

Allocation of resources example

CROPS	Potatoes	Wheat	RESOURCES
REQUIRED LABOR (man-days/acre)	2	4/5	180 man-days
REQUIRED ACREAGE (acres/acre)	1	1	120 acres
REQUIRED CAPITAL ($/acre)	0	10	1,000 dollars
RETURN REVENUE ($/acre)	40	80	

Mathematical Model

Maximize $R = 40y_1 + 80y_2$

subject to

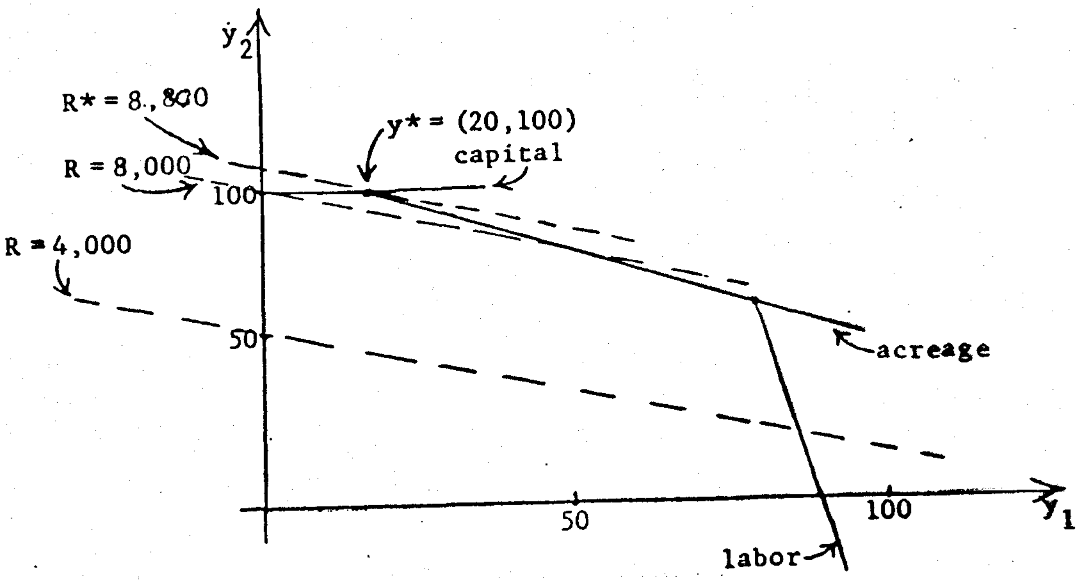
$$2y_1 + (4/5)y_2 \le 180 \qquad \text{(labor constraint)}$$

$$y_1 + y_2 \le 120 \qquad \text{(acreage constraint)}$$

$$10y_2 \le 1{,}000 \qquad \text{(capital constraint)}$$

$$y_1 \ge 0 \qquad y_2 \ge 0$$

Graphical representation



Terminology: The acreage and capital constraints are active (tight), while the labor constraint and the constraints $y_1 \geq 0$ and $y_2 \geq 0$ are inactive (slack).

Observations: When they exist, optimal solutions to nontrivial linear optimization problems always occur on the "boundary" of the feasible solution set. Moreover, if optimal solutions exist, then at least one optimal solution must occur at a "vertex" of the "polyhedral" feasible solution set; in which case only the vertices need to be examined to determine an optimal solution. Finally, note that the differential calculus can not be used to solve the preceding example problem.

Diet problem example

Nutrients	Bread	Butter	Milk	Requirement
vitamin A	2u/slice	4u/pat	2u/glass	13u
vitamin B	2u/slice	1u/pat	3u/glass	10u
Cost	2¢/slice	2¢/pat	10¢/glass	

At least two glasses of milk are also required, and no more than 2.5 slices of bread can be consumed.

Mathematical Model

Minimize $C = 2y_1 + 2y_2 + 10y_3$

subject to

$$2y_1 + 4y_2 + 2y_3 \ge 13$$

$$2y_1 + y_2 + 3y_3 \ge 10$$

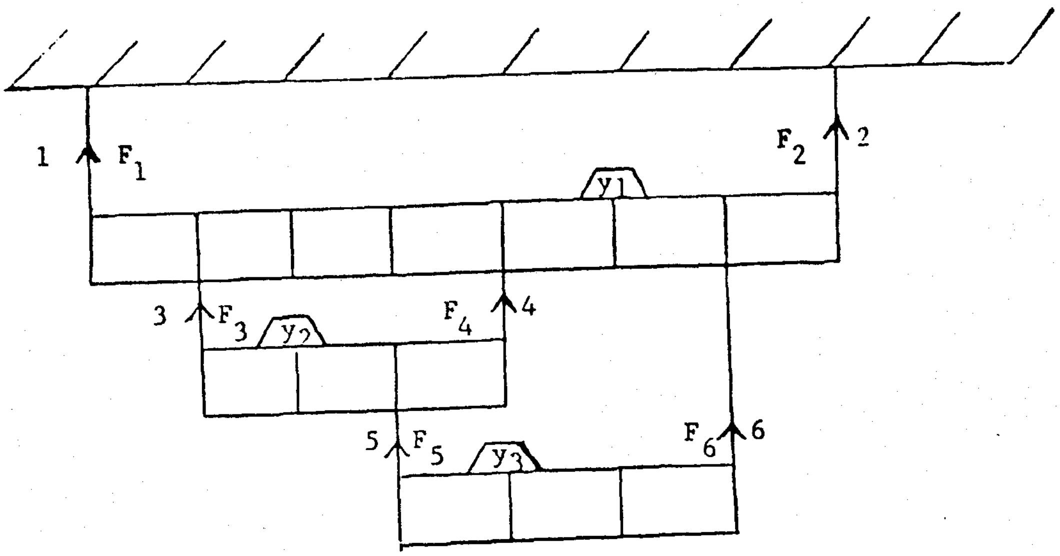
$$y_3 \ge 2$$

$$y_1 \le \frac{5}{2}$$

$$y_1 \ge 0, \ y_2 \ge 0, \ y_3 \ge 0$$

Exercise: Given that $y_3 = 2$, solve the resulting diet problem graphically.

Scaffold problem example



t_i is maximum tension permitted in wire i, $i = 1, 2, \ldots, 6$

Mathematical Model

$$\begin{array}{l}\text{force}\\ \text{balance}\\ \text{equations}\end{array}\left\{\begin{array}{rcl} F_1+F_2-F_3-F_4 \qquad -F_6 &=& y_1 \\ F_3+F_4-F_5 &=& y_2 \\ F_5+F_6 &=& y_3 \end{array}\right.$$

$$\begin{array}{l}\text{torque}\\ \text{balance}\\ \text{equations}\end{array}\left\{\begin{array}{rcl} 7F_2-F_3-4F_4 \qquad -6F_6 &=& 5y_1 \\ +3F_4-2F_5 &=& y_2 \\ +3F_6 &=& y_3 \end{array}\right.$$

$$\text{solution}\left\{\begin{array}{l} F_1 = \frac{2}{7}y_1 + \frac{5}{7}y_2 + \frac{3}{7}y_3 \\ F_2 = \frac{5}{7}y_1 + \frac{2}{7}y_2 + \frac{4}{7}y_3 \\ F_3 = \qquad \frac{2}{3}y_2 + \frac{2}{9}y_3 \\ F_4 = \qquad \frac{1}{3}y_2 + \frac{4}{9}y_3 \\ F_5 = \qquad\qquad \frac{2}{3}y_3 \\ F_6 = \qquad\qquad \frac{1}{3}y_3 \end{array}\right.$$

Maximize $y_1 + y_2 + y_3$ subject to

$$
\begin{aligned}
\tfrac{2}{7} y_1 + \tfrac{5}{7} y_2 + \tfrac{3}{7} y_3 &\le t_1 \\
\tfrac{5}{7} y_1 + \tfrac{2}{7} y_2 + \tfrac{4}{7} y_3 &\le t_2 \\
\tfrac{2}{3} y_2 + \tfrac{2}{9} y_3 &\le t_3 \\
\tfrac{1}{3} y_2 + \tfrac{4}{9} y_3 &\le t_4 \\
\tfrac{2}{3} y_3 &\le t_5 \\
\tfrac{1}{3} y_3 &\le t_6
\end{aligned}
$$

$$y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$$

<u>Exercise</u>: Given that $y_3 = 0$ and that $t_1 = t_2 = 300$, $t_3 = t_4 = 100$, $t_5 = t_6 = 50$, solve the resulting scaffold problem graphically.

Transportation Problem

$r_1 \quad r_2 \quad r_j \quad r_n$ Requirements

$\times \quad \times \; \cdots \; \times \; \cdots \; \times$

$R_1 \quad R_2 \quad R_j \quad R_n$ Retail outlets

$s_1 \quad s_2 \quad s_i \quad s_m$ Supplies

$\otimes \quad \otimes \; \cdots \; \otimes \; \cdots \; \otimes$

$W_1 \quad W_2 \quad W_i \quad W_m$ Warehouses

c_{ij} is the unit shipping cost from W_i to R_j

Mathematical Model

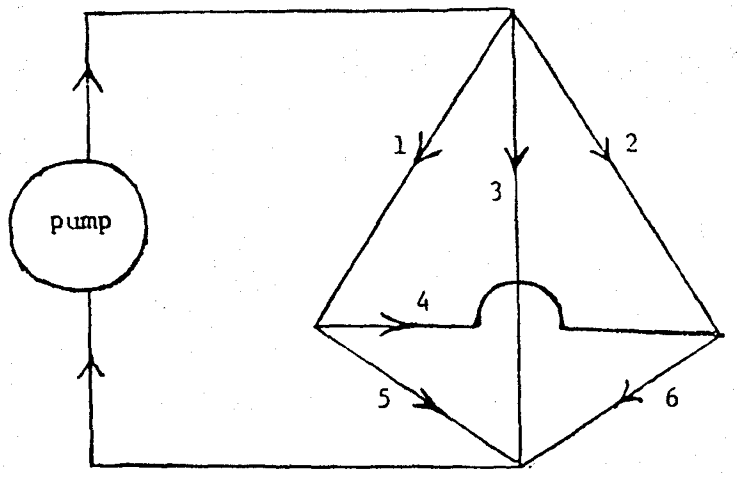
$$\text{Minimize} \quad \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} \, x_{ij} \quad \text{subject to}$$

$$\sum_{j=1}^{n} x_{ij} \le s_i \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = r_j \qquad j = 1, 2, \ldots, n$$

$$x_{ij} \ge 0$$

Network flow example



Link i has capacity
c_i, $i = 1, 2, \ldots, 6$

Mathematical model

Maximize s subject to

$$
\begin{aligned}
s &= x_1 + x_2 + x_3 \\
0 &= -x_1 + x_4 + x_5 \\
0 &= -x_2 - x_4 + x_6 \\
s &= x_3 + x_5 + x_6
\end{aligned}
$$

$$-c_i \le x_i \le c_i, \quad i = 1, 2, \ldots, 6$$

FUNDAMENTAL CONCEPTS AND TERMINOLOGY

An *optimization problem* (f,S) consists of a given *objective function* f and a given *feasible solution set* S, with f assuming real values $f(x)$ on its domain D and with S being a subset of D. If S is a proper subset of D, the elements of S are specified by additional conditions -- typically in the form of *constraints* on other functions termed *constraint functions*. A given constraint function u usually has the same domain D as f, and its values $u(x)$ are normally constrained via either $\le$, $\ge$ or $=$ relative to a given constant b. Constraints of the type $u(x) \le b$ or $u(x) \ge b$ are termed *inequality constraints*, while constraints of the type $u(x) = b$ are termed *equality constraints*. (Constraints involving $<$ or $>$ are normally not considered.)

An optimization problem (f,S) is either *inconsistent* (sometimes termed "infeasible") or *consistent* (sometimes termed "feasible"), depending respectively on whether its feasible solution set S is empty or nonempty. In particular, a consistent problem (f,S) has at least one *feasible solution* $x \in S$, and each such feasible solution x provides a *feasible value* $f(x)$ in the *feasible value set* $\{f(x) | x \in S\}$.

An optimization problem (f,S) is either a *maximization problem* or a *minimization problem*, depending respectively on whether the feasible value $f(x)$ is to be made as large as possible or as small as possible. A consistent problem (f,S) is either *unbounded* or *bounded*, depending respectively on whether $\{f(x) | x \in S\}$ is unbounded or bounded in the desired direction (namely, the positive direction for maximization problems, and the negative direction for minimization problems). In particular, a bounded problem (f,S) has an *optimal value* f^* that is either the "least upper bound" or the "greatest lower bound" of $\{f(x) | x \in S\}$, depending respectively on whether (f,S) is a maximization problem or a minimization problem. This optimal value f^* is either the maximum value of (f,S) or the minimum value of (f,S), depending respectively on whether (f,S) is a maximization problem or a minimization problem. In either case, (f,S) may, or may not, have an optimal solution x^*, namely, a feasible solution $x \in S$ for which $f(x) = f^*$. In both cases, the optimal solution set S^* for (f,S) consists of all such solutions x^* (if there are any).

An optimization problem (f,S) is a linear optimization problem if its objective function f is the sum of a linear function and a constant (on $D = R^n$) and if its constraint functions u are linear and finite in number (on $D = R^n$). Although the sum of a linear function and a constant is usually called an "affine function," the term "affine optimization" is almost never used.

The next section shows how to formulate a given linear optimization problem so that each of its inequality constraints has the "simple form" $x_j \geq 0$. The simplicity of these inequality constraints is the key to solving and analyzing linear optimization problems via certain extensions of the powerful algorithms and theory from linear algebra and matrix theory.

Standard Formulations and Equivalent Problems

Each inequality constraint $u(x) \geq b$ not of the simple form $x_j \geq 0$ can be replaced by one of simple form along with an equality constraint -- simply by introducing a slack variable $s \geq 0$ along with the equality constraint $u(x) - s = b$. Similarly, each inequality constraint $u(x) \leq b$ not of the simple form $-x_j \leq 0$ can be replaced by one of simple form along with an equality constraint -- simply by introducing a slack variable $t \geq 0$ along with the equality constraint $u(x) + t = b$. Introducing a different slack variable into each inequality constraint not of simple form produces a standard formulation --

namely, a linear optimization problem involving only equality constraints and/or inequality constraints of simple form.

For example, in the allocation of resources problem, the introduction of slack variables y_3, y_4 and y_5 produces the standard formulation

$$\begin{aligned} &\text{Maximize} \quad R = 40y_1 + 80y_2 \qquad \text{subject to} \\ &2y_1 + (4/5)\,y_2 + y_3 = 180 \\ &y_1 + y_2 + y_4 = 120 \\ &10y_2 + y_5 = 1{,}000 \\ &y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0,\ y_5 \geq 0 \end{aligned}$$

This is a standard formulation because each of its constraints is either an equality constraint or an inequality constraint of simple form.

Note that the slack variables y_3, y_4, y_5 have important economic interpretations -- with y_3 being the amount of excess labor, while y_4 is the amount of excess acreage, and y_5 is the amount of excess capital. In fact, the original formulation could have incorporated these variables directly (eliminating the necessity for introducing them here as a purely mathematical artifice).

In veiw of this economic equivalence between the original formulation and the preceding standard formulation, it should not be surprising that they are mathematically equivalent--in the sense that their maximum values are the same, and there is a direct correspondence between the elements in their respective optimal solution sets. Of course, this equivalence means that an optimal solution to either formulation provides an optimal solution to the other formulation; so we can solve the formulation that is mathematically most tractable (in the case of more complicated models, usually the standard formulation).

Other reformulations and transformations (not necessarily involving the introduction of slack variables) are needed in linear optimization. To treat all such cases, we need a rather general definition of equivalence. In particular, two maximization problems (or two minimization problems) are equivalent if

1. There is a transformation that "maps" the feasible solution set for one problem "onto" the feasible solution set for the other problem (not necessarily in a "one-to-one" fashion),
2. The respective objective values for each feasible solution and its "image" are the same.

(To avoid confusion with other concepts of "equivalence", it is worth noting that this relation between optimization problems is actually not an "equivalence relation"--in that it is not "symmetric", because the transformations need not be one-to-one).

As a concrete example, the original formulation and the preceding standard formulation of the allocation of resources problem are equivalent because

1. The transformation $(y_1,y_2,y_3,y_4,y_5) \rightarrow (y_1,y_2)$ clearly maps the set of all feasible solutions (y_1,y_2,y_3,y_4,y_5) to the standard formulation onto the set of all feasible solutions (y_1,y_2) to the original formulation (actually in one-to-one fashion in this case, because the deleted slack components y_3,y_4,y_5 are,

of course, uniquely determined from the nondeleted components y_1, y_2 by the constraint equations),

2. The respective objective values for each feasible solution (y_1,y_2,y_3,y_4,y_5) and its image (y_1,y_2) are the same, namely $40y_1 + 80y_2$.

In the general case, the image of a given optimal solution must also be an optimal solution; otherwise, its feasibility (by property 1) would imply that its objective value can be made larger (in the maximization case), which would then imply (by property 1 and property 2) that the given optimal solution is not optimal--a contradiction. Conversely, each preimage of a given optimal solution must also be an optimal solution; otherwise, essentially the same argument leads to another contradiction. Consequently, it is impossible for only one problem from an equivalent pair to have optimal solutions; that is, <u>either both problems from an equivalent pair have optimal solutions, or neither problem has optimal solutions</u>. Moreover, <u>if both problems from an equivalent pair have optimal solutions, their respective maximum values (or minimum values) are the same, and the transformation that maps one feasible solution set onto the other feasible solution set also maps the former optimal solution set onto the latter optimal solution set</u> (simply by "restricting" the "domain" and "range" of the transformation from the respective feasible solution sets to the respective optimal solution sets). In particular then, <u>solving the problem that is mathematically most tractable automatically solves the other problem in an equivalent pair</u>.

Continuing with our concrete example, we can now assert that an optimal solution $(y_1^*,y_2^*,y_3^*,y_4^*,y_5^*)$ to the standard formulation provides an optimal solution (y_1^*,y_2^*) to the original formulation (simply by deleting the slack components), while an optimal solution (y_1^*,y_2^*) to the original formulation provides an optimal solution $(y_1^*,y_2^*,y_3^*,y_4^*,y_5^*)$ to the standard formulation, with the slack components y_3^*,y_4^*,y_5^* determined from the original components y_1^*,y_2^* by the constraint equations.

Throughout the rest of these notes, a complete verification that a given problem is equivalent to the various formulations constructed from it is usually left to the reader (without explicitly saying so).

Exercises: 1. Place the diet problem (with all three variables) in standard form by introducing appropriate slack variables, and then interpret the slack variables.

2. Place the scaffold problem (with all three variables) in standard form by introducing appropriate slack variables, and then interpret the slack variables.

3. Show that each linear optimization problem with at least one equality constraint of the form $x_j \leq 0$ can be reformulated as an equivalent linear optimization problem with no such contraints and with the same number of variables. (Hint: Consider the transformation that replaces each variable x_j in a constraint $x_j \leq 0$ by a new variable $x_j' \triangleq -x_j$.)

Problems in Canonical Form

The main algorithm for solving linear optimization problems (the "simplex algorithm") requires that they first be placed in canonical form -- a standard form in which: (i) each variable $x_i \geq 0$, and (ii) each equality constraint includes a variable x_i, termed a basic variable (or "dependent variable"), that is multiplied by plus one and appears in neither the objective function nor any other equality constraint.

Note that the original formulation of the allocation of resources problem is not in canonical form (because it is not in standard form). However, the equivalent standard formulation given in the preceding section is obviously in canonical form, with the slack variables y_3, y_4 and y_5 serving as the basic variables. (We shall see that slack variables and basic variables are not always the same; so our terminology is not redundant.)

Other examples of problems in canonical form are:

$$\text{Min } f = \qquad -x_3 + 2x_4 - 3x_5 \qquad \text{subject to}$$

$$\begin{aligned} x_1 \qquad\quad +2x_3 - x_4 + x_5 &= -1 \\ x_2 - 3x_3 + 5x_4 - 2x_5 &= 3 \end{aligned}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

$$\begin{array}{lrrrrrrl} \text{Max } f = & -x_1 & & +2x_3 & & & & \text{subject to} \\ & -x_1 & & -4x_3 & +x_4 & & = & -2 \\ & 2x_1 & & +x_3 & & +x_5 & = & 0 \\ & -2x_1 & +x_2 & -x_3 & & & = & 1 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

$$\begin{array}{lrrrrrrl} \text{Max } f = & -x_1 & +2x_2 & & & & & \text{subject to} \\ & -x_1 & -4x_2 & +x_3 & & & = & -2 \\ & 2x_1 & +x_2 & & +x_4 & & = & 0 \\ & -2x_1 & -x_2 & & & +x_5 & = & 1 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

All three examples are clearly in standard form. In the first example, equality constraint one includes a basic variable x_1 while equality constraint two includes a (necessarily different) basic variable x_2. In the second example, equality constraint one includes a basic variable x_4 while equality constraint two includes a basic variable x_5 and equality constraint three includes a basic variable x_2 (with each such variable being necessarily different). In the third example, equality constraints one through three include the basic variables x_3, x_4 and x_5, respectively.

If a given equality constraint has more than one basic variable x_i, all such basic variables x_i can obviously be grouped into a sum that can then be replaced by a single new variable x_k (because all such basic variables x_i appear in neither the objective function nor any other equality constraint). Clearly, the new variable x_k becomes the only basic variable in the given equality constraint, and x_k is to be restricted by the constraint $x_k \geq 0$.

Carrying out such a replacement (or change of variables) for each equality constraint that has more than one basic variable x_i obviously leads to an equivalent canonical form in which the basic variables are in a one-to-one correspondence with the equality constraints. In particular then, there is no loss of generality in subsequently assuming that a problem in canonical form has the same number of basic variables as equality constraints.

Each variable that is not basic is said to be a nonbasic variable (or "independent variable"). In the first example (of this section), variables x_3, x_4 and x_5 are nonbasic. In the second examble, variables x_1 and x_3 are nonbasic.

It is important to note that, in essence, the objective equation and the equality constraint equations make the objective variable f and the basic variables x_i functions of only the nonbasic variables x_j (a fact that justifies the alternative terminology).

The canonicality of the first and third examples is readily visualized because the coefficient matrix associated with the equality constraints contains as a submatrix the identity matrix with the same number of rows. Of course, each problem that is already in canonical form but has no such submatrix can be reformulated with such a submatrix, simply by permuting appropriate variables (and their corresponding coefficient matrix columns). In particular, note that such a permutation for the second example followed by an appropriate relabeling of the variables produces the third example.

For notational convenience, we will frequently arrange the variables so that the basic variables are enumerated either first (as in the first example) or last (as in the last example).

Exercise: 1. For each of the following problems that is in canonical form, state the basic variables and then the nonbasic variables

(a) Min $f = 2x_1 + 3x_2 - x_3$ subject to

$$-3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$2x_1 - x_2 + 3x_3 + x_5 = 4$$

$$x_1 \geq 0 \qquad x_4 \geq 0$$

(b) Min $f = 2x_1 + 3x_2 - x_3 + 2x_4$ subject to

$$-3x_1 + 2x_2 + 4x_3 + x_4 = 1$$

$$2x_1 - x_2 + 3x_3 + x_5 = 4$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

(c) Max $f = x_1 + x_3 - 2x_4$ subject to

$$-2x_1 - 3x_3 + 2x_4 + x_5 = 3$$

$$3x_1 + x_2 - x_3 + x_4 = 2$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

Can you place in canonical form those problems that are not already in canonical form? (Hint: Dependent variables can always be eliminated in favor of independent variables.)

2. Place the diet problem in canonical form by manipulating its standard formulation. (Hint: multiplying each side of an equation by minus one produces an equivalent equation -- one with the same solution set.)

3. Place the scaffold problem in canonical form.