Linear Conservation Laws of Nonholonomic Systems with Symmetry

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Abstract. In nonholonomic dynamics, symmetries do not always lead to conservation laws. In this paper we study conditions for a nonholonomic system with symmetry to have conservation laws linear in momentum.

1 Introduction

Nonholonomic systems are mechanical systems subject to constraints imposed on *velocities* opposed to *holonomic* systems where the constraints are *positional*. This classification goes back to Hertz [9]. In this paper we study conditions for existence of conservation laws of nonholonomic systems with symmetries that are linear in momentum. These conservation laws, when present, are useful in the study of the qualitative properties of constrained dynamics and in stability analysis.

The structure of linear conservation laws in holonomic mechanics is well understood and is summarized in the Noether theorem (see Noether [16] and Marsden [14]): *The projections of the conjugate momentum of a system with symmetry along the generators of the symmetry group action are preserved by the flow.* This statement is not in general valid for nonholonomic systems; as discussed in Bloch, Krishnaprasad, Marsden, and Murray [3], the *nonholonomic momentum,* which is the collection of the components of the momentum along the directions tangent to the group orbit and consistent with the constraints, typically is not preserved by the flow but instead satisfies the *momentum equation.* The latter produces momentum conservation laws in the case of *horizontal symmetries* (see [3] for details).

The nonholonomic Noether theorem is discussed in Cushman, Kemppainen, Śniatycki, and Bates [7] using the Hamiltonian approach. The question of when the presence of symmetry leads to momentum conservation laws is studied in [7] for one-dimensional groups only. In Śniatycki [22] this theory is extended to the case of systems with symmetries subject to nonlinear nonholonomic constraints.

Agostinelli [1] gives conditions for existence of a linear conservation law of a nonholonomic system. These conditions are represented by a system of linear partial differential equations for the coefficients of the conservation law. In this paper we pose a different question: *Given a nonholonomic system with symmetry, find when the group action produces linear conservation laws.* Our main

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observation is that for some nonholonomic systems it is possible to find vector fields that are *not* the generators of the group action but nevertheless identify the directions in which the momentum is preserved. We emphasize that such vector fields *are* assumed to be the generators in [22]. We study the conditions for existence of such vector fields; these fields include the horizontal symmetries as a special case. Our result thus gives conditions for existence of solutions of partial differential equations obtained by Agostinelli [1].

The paper is organized as follows: In section 2 we give a brief overview of nonholonomic dynamics. In section 3 we discuss the classical Noether theorem and demonstrate that it may fail in the presence of nonholonomic constraints. The main results are exposed in section 4. We first discuss how the momentum equation can lead to linear conservation laws in the pure transport case. We then consider a more general case in which the quadratic momentum terms are present in the momentum equation and obtain the conditions for existence of linear conservation laws. Further analysis of the properties of these conservation laws in the case of non-free group action as well as their role in studying the measure-preserving constrained flows and discrete nonholonomic dynamics will appear in future publications.

2 An Overview of Nonholonomic Dynamics

In this section we briefly discuss the dynamics of nonholonomic systems with symmetries. We refer the reader to Bloch, Krishnaprasad, Marsden, and Murray [3] and Zenkov, Bloch, and Marsden [27] for a more complete exposition.

2.1 The Lagrange–d'Alembert Principle

A nonholonomic system is a mechanical system subject to constraints imposed on *velocities.* We confine our attention to nonholonomic constraints that are linear and homogeneous in the velocity components. Accordingly, we consider a configuration space Q and a distribution $\mathcal{D}$ that describes these constraints. Recall that a distribution $\mathcal{D}$ is a collection of linear subspaces of the tangent spaces of Q; we denote these spaces by $\mathcal{D}_q \subset T_qQ$, one for each $q \in Q$. Constraints are nonholonomic if and only if the distribution $\mathcal{D}$ is nonintegrable. A curve $q(t) \in Q$ will be said to *satisfy the constraints* if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t.

Consider a Lagrangian $L : TQ \to \mathbb{R}$. In coordinates $q^i, i = 1, \dots, n$, on Q with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we write $L(q^i, \dot{q}^i)$. The equations of motion are given by the following Lagrange–d'Alembert principle: *The Lagrange–d'Alembert equations of motion for the system are those determined by*

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t) \in \mathcal{D}_{q(t)}$ for each t where $a \le t \le b$. This principle is supplemented by the condition that the curve itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations (see Bloch, Krishnaprasad, Marsden, and Murray [3] for a discussion and references).

2.2 The Geometry of Nonholonomic Systems with Symmetry

Let G be a Lie group that acts freely and properly on the configuration space Q. We say that G is a *symmetry group* of the nonholonomic system if both the Lagrangian L and distribution $\mathcal{D}$ are invariant under the induced action of G on the tangent bundle TQ. The manifold Q/G is called the *shape space* of the system and the configuration space has the structure of a principal fiber bundle $\pi : Q \to Q/G$.

Orbits and Shape Space. The group orbit through a point q, an (immersed) submanifold, is denoted

$$\mathrm{Orb}(q) := \{gq \mid g \in G\}.$$

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G. For an element $\xi \in \mathfrak{g}$, we write ξ_Q, a vector field on Q for the corresponding infinitesimal generator, so these are also the tangent spaces to the group orbits. Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in both $\mathcal{D}_q$ and $T_q(\mathrm{Orb}(q))$:

$$\mathfrak{g}^q := \{\xi \in \mathfrak{g} \mid \xi_Q(q) \in \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))\}\,.$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$ is denoted by $\mathfrak{g}^{\mathcal{D}}$.

Reduced Dynamics. Assuming that the Lagrangian and the constraint distribution are G-invariant, we can form the *reduced velocity phase space* TQ/G and the *reduced constraint space* $\mathcal{D}/G$. The Lagrangian L induces well defined functions, the *reduced Lagrangian* $l : TQ/G \to \mathbb{R}$ and the *constrained reduced Lagrangian* $l_c : \mathcal{D}/G \to \mathbb{R}$, satisfying $L = l \circ \pi_{TQ}$ and $L|_{\mathcal{D}} = l_c \circ \pi_{\mathcal{D}}$ where $\pi_{TQ} : TQ \to TQ/G$ and $\pi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}/G$ are the projections. The vector field on the manifold $\mathcal{D}$ determined by the Lagrange–d'Alembert equations (including the constraints) is G-invariant, and so defines a reduced vector field on the quotient manifold $\mathcal{D}/G$. Following Cendra *et al.* [5], we call these equations the *Lagrange–d'Alembert–Poincaré equations.*

Let a local trivialization be chosen on the principle bundle $\pi : Q \to Q/G$, with a local representation having components denoted (r, g). Let r, an element of the shape space Q/G, have coordinates denoted r^α, $\alpha = 1, \ldots, \sigma$, and let g be group variables for the fiber, G. In the above, $\sigma = \dim Q/G$. In such a representation, the group action is the left action of G on the second factor. The coordinates (r, g) induce the coordinates $(r, \dot{r}, \xi)$ on TQ/G, where $\xi = g^{-1}\dot{g}$. The full system of equations of motion consists of the following two subsystems:

1. The Lagrange–d'Alembert–Poincaré equation on $\mathcal{D}/G$ (see theorem 1).

2. The *reconstruction equation* $\dot{g} = g\xi$.

The Nonholonomic Momentum in Body Representation. The Lagrangian L is invariant under the left action of G and so it depends on g and $\dot{g}$ only through the combination $\xi = g^{-1}\dot{g}$. Thus the reduced Lagrangian l is given by

$$l(r, \dot{r}, \xi) = L(r, g, \dot{r}, \dot{g}).$$

Choose a q-dependent basis $e_A(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$ in the following way. First, one chooses, for each r, such a basis at the identity element $g = \mathrm{Id}$, say

$$e_1(r), e_2(r), \ldots, e_m(r), e_{m+1}(r), \ldots, e_k(r), \qquad k = \dim G.$$

It is assumed in this paper that the subspaces

$$\mathrm{span}\{e_1(r), \ldots, e_m(r)\} \quad \text{and} \quad \mathrm{span}\{e_{m+1}(r), \ldots, e_k(r)\}$$

are orthogonal in the kinetic energy metric. Now define the *body fixed basis* by

$$e_A(r, g) = \mathrm{Ad}_g\, e_A(r), \qquad A = 1, \ldots, k.$$

Then the first m elements will indeed span the subspace $\mathfrak{g}^q$ since the distribution is invariant. We denote the structure constants of the Lie algebra relative to this basis by C^C_{AB}.

Assume that the Lagrangian has the form of kinetic minus potential energy, and that the constraints and the orbit directions span the entire tangent space to the configuration space. Then it is possible to introduce a new Lie algebra variable Ω called the *body angular velocity* such that:

1. $\Omega = \mathcal{A}\dot{r} + \xi$, where the Lie algebra valued form $\mathcal{A} = \mathcal{A}^A_\alpha e_A(r) dr^\alpha$ is called the *nonholonomic connection* (see Bloch, Krishnaprasad, Marsden, and Murray [3] for details).
2. The constraints are given by $\Omega \in \mathrm{span}\{e_1(r), \ldots, e_m(r)\}$ or $\Omega^{m+1} = \cdots = \Omega^k = 0$.
3. The reduced Lagrangian in the variables $(r, \dot{r}, \Omega)$ becomes

$$l(r^\alpha, \dot{r}^\alpha, \Omega^A) = \frac{1}{2} g_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + \frac{1}{2}\mathbb{I}_{AB}\Omega^A\Omega^B + \lambda_{a'\alpha}\dot{r}^\alpha\Omega^{a'} - U(r).$$

Here $g_{\alpha\beta}$ are coefficients of the kinetic energy metric induced on the manifold Q/G, $\mathbb{I}_{AB}$ are components of the *locked inertia tensor* defined by

$$\langle \mathbb{I}(r)\xi, \eta\rangle = \langle\langle \xi_Q, \eta_Q \rangle\rangle, \qquad \xi, \eta \in \mathfrak{g},$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the kinetic energy metric, and $U(r)$ is the potential energy. The constrained reduced Lagrangian becomes especially simple in the variables $(r, \dot{r}, \Omega)$:

$$l_c = \frac{1}{2} g_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + \frac{1}{2}\mathbb{I}_{ac}\Omega^a\Omega^c - U, \qquad a, c = 1, \ldots, m. \tag{1}$$

We remark that this choice of Ω block-diagonalizes the reduced constrained kinetic energy, *i.e.,* it eliminates the terms proportional to $\Omega^a\dot{r}^\alpha$ in (1).

The *nonholonomic momentum in body representation* is defined by

$$p_a = \frac{\partial l}{\partial \Omega^a} = \frac{\partial l_c}{\partial \Omega^a}, \qquad a = 1, \ldots, m.$$

Notice that the nonholonomic momentum may be viewed as a collection of components of the ordinary momentum map along the constraint directions.

The Lagrange–d'Alembert–Poincaré Equations. As in Bloch *et al.* [3], the reduced equations of motion are given by the next theorem.

Theorem 1. *The following* ***reduced nonholonomic Lagrange–d'Alembert– Poincaré equations*** *hold for each* $1 \le \alpha \le \sigma$ *and* $1 \le a \le m$*:*

$$\frac{d}{dt}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} = -\frac{\partial I^{cd}}{\partial r^\alpha} p_c p_d - \mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot{r}^\beta - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma, \tag{2}$$

$$\dot{p}_a = C^c_{ba} I^{bd} p_c p_d + \mathcal{D}^c_{a\alpha} p_c \dot{r}^\alpha + \mathcal{D}_{\alpha\beta a} \dot{r}^\alpha \dot{r}^\beta. \tag{3}$$

Here I^{ad} *are the components of the inverse locked inertia tensor; and the coefficients* $\mathcal{B}^C_{\alpha\beta}$ $\mathcal{D}^c_{b\alpha}$*,* $\mathcal{D}_{\alpha\beta b}$*,* $\mathcal{K}_{\alpha\beta\gamma}$ *are given by the formulae*

$$\begin{aligned}
\mathcal{B}^C_{\alpha\beta} &= \frac{\partial \mathcal{A}^C_\alpha}{\partial r^\beta} - \frac{\partial \mathcal{A}^C_\beta}{\partial r^\alpha} - C^C_{BA} \mathcal{A}^A_\alpha \mathcal{A}^B_\beta + \gamma^C_{A\beta} \mathcal{A}^A_\alpha - \gamma^C_{A\alpha} \mathcal{A}^A_\beta, \\
\mathcal{D}^c_{b\alpha} &= -C^c_{Ab} \mathcal{A}^A_\alpha + \gamma^c_{b\alpha} + \lambda_{a'\alpha} C^{a'}_{ab} I^{ac}, \\
\mathcal{D}_{\alpha\beta b} &= \lambda_{a'\alpha} \left(\gamma^{a'}_{b\beta} - C^{a'}_{Bb} \mathcal{A}^B_\beta \right), \\
\mathcal{K}_{\alpha\beta\gamma} &= \lambda_{a'\gamma} \mathcal{B}^{a'}_{\alpha\beta},
\end{aligned}$$

where the coefficients $\gamma^C_{B\alpha}$ *are defined by*

$$\frac{\partial e_B}{\partial r^\alpha} = \gamma^C_{B\alpha} e_C.$$

3 The Noether Theorem

In this section we state the Noether theorem for holonomic systems and present examples that show how this theorem may fail in the presence of nonholonomic constraints.

3.1 The Noether Theorem

Let L be a G-invariant Lagrangian. Analytically this means that $L(r, \dot{r}, g, \dot{g}) = l(r, \dot{r}, \xi)$, where $\xi = g^{-1}\dot{g}$ and (r, g) are the bundle coordinates. The *momentum map* $J : TQ \to \mathfrak{g}^*$ is defined by

$$\langle J, \eta \rangle = \frac{\partial L}{\partial \dot{q}^i} \eta^i_Q, \qquad \eta \in \mathfrak{g},$$

i.e., it is the collection of components of the *fiber derivative* of the Lagrangian that are aligned along the symmetry directions. As usual, G_η denotes the subgroup of the Lie group G generated by the Lie algebra element η. The generators of the action $G \times Q \to Q$ are denoted η_Q.

Theorem 2. (Noether) *For each one-dimensional subgroup* G_η *of* G*, the component* $\langle J, \eta \rangle$ *of the momentum is conserved.*

See Noether [16] and Marsden [14] for details.

The Noether theorem thus states that for holonomic systems the projections of the conjugate momentum along the generators of the symmetry group action are constants of motion.

3.2 Examples

The following examples show that theorem 2 may not hold for certain nonholonomic systems.

The Rattleback. A rattleback is a convex nonsymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameter values, and for other values, to exhibit multiple reversals. Basic references on the rattleback are Walker [24], Karapetyan [10, 11], Markeev [13], Pascal [18, 19], and Bondi [4].

Some of the steady-state rotations of the rattleback about the vertical line through the contact point are asymptotically stable with respect to some of the phase variables. Therefore, the nonholonomic momentum, which here equals the vertical component of the angular momentum of the body, is not a constant of motion.

It is known (see e.g. Zenkov, Bloch, and Marsden [27]) that this system has *local* invariant manifolds near the relative equilibria that correspond to the steady-state rotations. However, these manifolds cannot be represented as level sets of linear conservation laws.

The Chaplygin Sphere. This system consists of a sphere rolling without slipping on a horizontal plane. The center of mass of this sphere is at the geometric center, but the principal moments of inertia are distinct. Chaplygin [6] proved integrability of this problem. Modern references for the Chaplygin sphere include Kozlov [12] and Schneider [21]. This system is $SE(2)$-invariant. The nonholonomic momentum map has just one component—the vertical component of the system's angular momentum—and is preserved.

The Rolling Disk. A classical example of a nonholonomic system that has linear integrals is a disk rolling without sliding on the xy-plane, as in Figure 1.

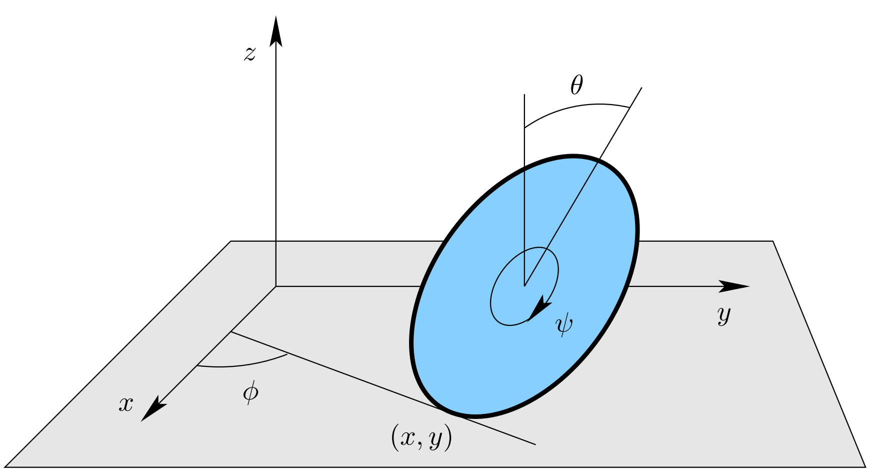

Figure 1: The geometry for the rolling disk.

As the figure indicates, we denote the coordinates of contact of the disk in the xy-plane by (x, y) and let θ, ϕ, and ψ denote the angle between the plane of the disk and the vertical axis, the "heading angle" of the disk, and the "self-rotation" angle of the disk, respectively.

A classical reference for the rolling disk is Vierkandt [23] who showed that on the reduced space $\mathcal{D}/SE(2)$—the constrained velocity phase space modulo the action of the Euclidean group $SE(2)$—most orbits of the system are periodic. Modern references that treat this example are Neimark and Fufaev [15], Hermans [8] and O'Reilly [17].

Denote the mass, the radius, and the moments of inertia of the disk by m, R, A, B, respectively. The Lagrangian is given by the kinetic minus potential energies:

$$L = \frac{m}{2}\left[(\xi - R(\dot\phi\sin\theta + \dot\psi))^2 + \eta^2\sin^2\theta + (\eta\cos\theta + R\dot\theta)^2\right] \\ + \frac{1}{2}\left[A(\dot\theta^2 + \dot\phi^2\cos^2\theta) + B(\dot\phi\sin\theta + \dot\psi)^2\right] - mgR\cos\theta,$$

where $\xi = \dot x\cos\phi + \dot y\sin\phi + R\dot\psi$ and $\eta = -\dot x\sin\phi + \dot y\cos\phi$, while the constraints are

$$\dot x = -\dot\psi R\cos\phi, \qquad \dot y = -\dot\psi R\sin\phi.$$

Note that the constraints may also be written as $\xi = 0$, $\eta = 0$.

The components of the nonholonomic momentum for the rolling disk are

$$p_1 = \frac{\partial L_c}{\partial\dot\phi}, \qquad p_2 = \frac{\partial L_c}{\partial\dot\psi},$$

where L_c is the constrained Lagrangian. One can check that p_1 and p_2 are not conserved throughout the motion but satisfy the following momentum equations:

$$\dot p_1 = mR^2\left(-\frac{\sin\theta}{A\cos\theta}\,p_1 + \left(\frac{\cos\theta}{mR^2+B} + \frac{\sin^2\theta}{A\cos\theta}\right)p_2\right)\dot\theta,$$
$$\dot p_2 = mR^2\left(-\frac{1}{A\cos\theta}\,p_1 + \frac{\sin\theta}{A\cos\theta}\,p_2\right)\dot\theta.$$

These equations define an integrable distribution

$$dp_1 = mR^2\left(-\frac{\sin\theta}{A\cos\theta}\,p_1 + \left(\frac{\cos\theta}{mR^2+B} + \frac{\sin^2\theta}{A\cos\theta}\right)p_2\right)d\theta,$$
$$dp_2 = mR^2\left(-\frac{1}{A\cos\theta}\,p_1 + \frac{\sin\theta}{A\cos\theta}\,p_2\right)d\theta.$$

The invariant leaves of this distribution

$$p_1 = c_1 p_1^1(\theta) + c_2 p_1^2(\theta), \qquad p_2 = c_1 p_2^1(\theta) + c_2 p_2^2(\theta)$$

define the conservation laws

$$\eta_1^1(\theta)p_1 + \eta_1^2(\theta)p_2 = c_1, \qquad \eta_2^1(\theta)p_1 + \eta_2^2(\theta)p_2 = c_2.$$

These conservation laws are linear in the components p_1 and p_2. However, these conservation laws are not represented by the projections of the conjugate momentum along the generators of the symmetry group action. Below we study the vector fields that replace these generators in the nonholonomic setting.

4 The Structure of the Momentum Equation and Conservation Laws

In this section we study invariant manifolds of certain classes of nonholonomic systems. We then show that these manifolds can be viewed as the level sets of conservation laws that are linear in the body momentum. The vector fields associated with these conservation laws are independent of the group variables.

4.1 The Pure Transport Case

The second group of terms in the momentum equation (3) defines the parallel transport equation in the appropriate fiber bundle:

$$\dot{p}_a = \mathcal{D}^b_{a\alpha} p_b \dot{r}^\alpha. \tag{4}$$

This equation defines the distribution

$$dp_a = \mathcal{D}^b_{a\alpha} p_b dr^\alpha. \tag{5}$$

Theorem 3. *The reduced Lagrange–d'Alembert–Poincaré equations* (2) *and* (3) *have a family of smooth invariant manifolds that are $\dot{r}$-invariant and project one-to-one onto $T(Q/G)$ if and only if the distribution* (5) *is integrable and the tensors $C^c_{ba} I^{bd}$ and $\mathcal{D}_{\alpha\beta a}$ are skew-symmetric in c, d and α, β, respectively.*

Proof. If the conditions of the theorem hold then distribution (5) is integrable. The skew-symmetry condition of the theorem implies that momentum equation (3) becomes equation (4), and thus each integral manifold of distribution (5) is flow-invariant and may be represented as the graph $p_a = P_a(r^\alpha, k_b)$, $a, b = 1, ..., m$, where k_b are constants. Here $P_a(r^\alpha, k_b)$ represent the family of solutions of equation (5). Therefore, the system (2) and (3) has a family of 2σ-dimensional invariant manifolds that are $\dot{r}$-invariant and project one-to-one onto $T(Q/G)$.

Suppose now that equations (2) and (3) have invariant manifolds that satisfy the statement of the theorem. Then these manifolds can be represented as level sets of the m conservation laws $p_a = P_a(r^\alpha, k_b)$. Differentiating these conservation laws along the vector field (2) and (3), we obtain

$$P_c(r,k) P_d(r,k) C^c_{ba} I^{bd} + \mathcal{D}^c_{a\alpha} P_c(r,k) \dot{r}^\alpha + \mathcal{D}_{\alpha\beta a} \dot{r}^\alpha \dot{r}^\beta = \frac{\partial P_a(r,k)}{\partial r^\alpha} \dot{r^\alpha}. \tag{6}$$

This condition holds for arbitrary values of r and k. The right-hand side of (6) is linear in $\dot{r}$. This implies skew-symmetry of tensors $C^c_{ba} I^{bd}$ and $\mathcal{D}_{\alpha\beta a}$. Equation (6) thus becomes

$$\mathcal{D}^c_{a\alpha} P_c(r,k) \dot{r}^\alpha = \frac{\partial P_a(r,k)}{\partial r^\alpha} \dot{r^\alpha}.$$

Therefore, the manifolds $p_a = P_a(r^\alpha, k_b)$ are integral manifolds of (5), which proves integrability of distribution (5). □

4.2 Properties of the Transport Distribution

We now prove that the space of integral manifolds of the distribution (5) is a vector space. Recall that m represents the number of unconstrained components of the system's angular velocity.

Theorem 4. *The space of integral manifolds of an integrable distribution* (5) *has the structure of a real m-dimensional vector space.*

Proof. If $p = P^1(r)$ and $p = P^2(r)$ are the integral manifolds of (5), then so is their linear combination $p = c_1P^1(r) + c_2P^2(r)$ for any real c_1 and c_2:

$$\begin{aligned} d(c_1P_a^1(r) + c_2P_a^2(r)) &= c_1dP_a^1(r) + c_2dP_a^2(r) \\ &= c_1\mathcal{D}_{a\alpha}^b P_b^1(r)dr^\alpha + c_2\mathcal{D}_{a\alpha}^b P_b^2(r)dr^\alpha = \mathcal{D}_{a\alpha}^b(c_1P_b^1(r) + c_2P_b^2(r))dr^\alpha. \end{aligned}$$

The space of integral manifolds is therefore a real vector space.

To prove that the dimension of this space equals m, we use the uniqueness property of distribution (5): For each (r_0, p_0) there exists a unique integral manifold of (5) through (r_0, p_0). Consider the following integral manifolds:

$$p_a = \mathcal{P}_a^1(r), \ldots, p_a = \mathcal{P}_a^m(r)$$

that satisfy

$$\mathcal{P}^1(r_0) = (1, 0, \ldots, 0), \quad \mathcal{P}^2(r_0) = (0, 1, \ldots, 0), \quad \mathcal{P}^m(r_0) = (0, 0, \ldots, 1).$$

Then for any integral manifold $p_a = P_a(r)$ satisfying $p_a^0 = P_a(r_0)$ there is a unique representation

$$P_a(r) = c_1\mathcal{P}_a^1(r) + \cdots + c_m\mathcal{P}_a^m(r).$$

Indeed, the equations

$$P_a(r_0) = c_1\mathcal{P}_a^1(r_0) + \cdots + c_m\mathcal{P}_a^m(r_0)$$

uniquely define the coefficients $c_1, \ldots, c_m$. The manifolds

$$p_a = P_a(r) \quad \text{and} \quad p_a = c_1\mathcal{P}_a^1(r) + \cdots + c_m\mathcal{P}_a^m(r)$$

then coincide because of the uniqueness property of the integral manifolds of (5). □

4.3 The Structure of the Conservation Laws in the Pure Transport Case

As discussed in §4.1, the integral manifolds of the transport connection (5) become the invariant manifolds of the system in the integrable pure transport case. We now show that these manifolds can be represented as the levels of linear conservation laws of the nonholonomic system.

Theorem 5. *The transport conservation laws $p_a = P_a(r, k)$ can be represented as $\langle p, \eta_a(r)\rangle = p_b\eta_a^b(r) = c_a$, $a = 1, \ldots, m$.*

Proof. Using the results of theorem 4, we rewrite the transport conservation laws as

$$p_a = \mathcal{P}_a^b(r) c_b. \tag{7}$$

The fields $\mathcal{P}^b(r)$, $b = 1, \dots, m$, are independent, and we thus are able to rewrite (7) as

$$p_b \eta_a^b(r) = c_a, \qquad a = 1, \dots, m,$$

where $\eta_a^b(r)$ are the elements of the inverse of $\mathcal{P}_a^b(r)$. □

4.4 Linear Conservation Laws in the Non-Pure Transport Case

Here we obtain the conditions for existence of linear integrals for systems whose momentum equation contains nontrivial non-transport terms.

Let $\langle p, \eta(r) \rangle = p_a \eta^a(r)$ be a linear conservation law of equations (2) and (3); we assume here that the tensor $\mathcal{D}_{\alpha\beta a}$ is skew-symmetric in α, β.

Theorem 6. *The reduced Lagrange–d'Alembert–Poincaré equations* (2) *and* (3) *have a conservation law* $\langle p, \eta(r) \rangle$ *if and only if*

(i) The distribution $dp_a = \mathcal{D}_{a\alpha}^b p_b dr^\alpha$ *is integrable;*

(ii) The vector field $\eta(r)$ *annihilates the tensor* $C_{ba}^c I^{bd} + C_{ba}^d I^{bc}$*:*

$$\left(C_{ba}^c I^{bd} + C_{ba}^d I^{bc} \right) \eta^a(r) = 0. \tag{8}$$

Proof. The flow derivative of $\langle p, \eta(r) \rangle$ is

$$\begin{aligned} \frac{d}{dt} \langle p, \eta(r) \rangle &= \dot{p}_a \eta^a(r) + p_a \dot{\eta}^a(r) \\ &= \left(\frac{\partial \eta^a}{\partial r^\alpha} p_a + \mathcal{D}_{a\alpha}^b p_b \eta^a \right) \dot{r}^\alpha + C_{ba}^c I^{bd} p_c p_d \eta^a. \end{aligned}$$

Hence, $\langle p, \eta(r) \rangle$ is a conservation law if and only if

$$\frac{\partial \eta^a}{\partial r^\alpha} p_a + \mathcal{D}_{a\alpha}^b p_b \eta^a = 0 \qquad \text{and} \qquad C_{ba}^c I^{bd} p_c p_d \eta^a = 0.$$

The last equality is equivalent to (8). Therefore, the necessary and sufficient condition for $\langle p, \eta(r) \rangle$ to be an integral becomes

$$d\eta^a = -\mathcal{D}_{b\alpha}^a \eta^b dr^\alpha \tag{9}$$

along with condition (ii) of the theorem.

The conservation law $\langle p, \eta(r) \rangle$ is a single-valued function if and only if $\eta(r)$ is a single-valued vector field; the latter is equivalent to integrability of the distribution (9). Define the operators ∇_α^* by

$$\nabla_\alpha^* = \frac{\partial}{\partial r^\alpha} - \mathcal{D}_{b\alpha}^a \eta^b \frac{\partial}{\partial \eta^a}.$$

The integrability condition for (9) is

$$\nabla^*_\alpha \nabla^*_\beta = \nabla^*_\beta \nabla^*_\alpha.$$

One can check by a direct computation that these operators commute if and only if

$$\frac{\partial \mathcal{D}^a_{b\alpha}}{r^\beta} - \frac{\partial \mathcal{D}^a_{b\beta}}{\partial r^\alpha} + \mathcal{D}^a_{c\beta}\mathcal{D}^c_{b\alpha} - \mathcal{D}^a_{c\alpha}\mathcal{D}^c_{b\beta} = 0.$$

This condition is equivalent to

$$\nabla_\alpha \nabla_\beta = \nabla_\beta \nabla_\alpha, \tag{10}$$

where

$$\nabla_\alpha = \frac{\partial}{\partial r^\alpha} + \mathcal{D}^a_{b\alpha} p_a \frac{\partial}{\partial p_b}$$

are the horizontal derivatives determined by the transport distribution (5). To finish the proof, we note that (10) is equivalent to condition (i) of the theorem. □

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