Abstract—The method of controlled Lagrangians for discrete mechanical systems is extended to the problem of stabilization of the rotations of a spacecraft with a symmetric rotor. In particular, stabilization about its intermediate axis of inertia is considered. The Moser–Veselov discretization is used to obtain the discrete dynamics of the system. Stabilization conditions for the continuous model and its discretization are compared. It is shown that stability of the discrete system is sufficient for stability of its continuous counterpart but not vice versa.

I. INTRODUCTION

The method of controlled Lagrangians for stabilization of relative equilibria (steady state motions) of mechanical systems originated in [8] and was then developed in Auckly, Kapitanski, and White [1], Bloch, Leonard, and Marsden [8], [9], [10], [11], [12], [3], [13], and Hamberg [21], [22]. A similar approach for Hamiltonian controlled systems was introduced and further studied in the work of Blankenstein, Ortega, van der Schaft, Maschke, Spong, and their collaborators (see, e.g., [28] and related references). The two methods were shown to be equivalent in [18] and a nonholonomic version was developed in [33], [34], and [2]. General approaches to the existence of a controlled Lagrangians and energy shaping are discussed in [16], [17] and [19] for example.

In a nutshell, the original controlled system is replaced with a new, uncontrolled Lagrangian system for a suitable new Lagrangian. The energy associated with this new Lagrangian is designed to be positive- or negative-definite at the (relative) equilibrium to be stabilized, and is then used as a Lyapunov function. The time-invariant feedback control law is obtained from the equivalence requirement for the new and old systems of equations of motion. If asymptotic stabilization is desired, dissipation-emulating terms are added to the control input.

Recall that the dynamics of mechanical systems (either unconstrained or subject to position constraints) can be derived from Hamilton’s principle. Discrete mechanical systems are obtained from their continuous-time counterparts by discretizing Hamilton’s principle, that is, by approximating the action integral of the continuous-time mechanics by an action sum followed by equating the variation of the latter to zero. The resulting discrete dynamics is known to be symplectic and, for systems with symmetry, momentum-preserving.

The method of controlled Lagrangians for discrete mechanical systems was introduced in Bloch, Leok, Marsden, and Zenkov [5], [6], [7]. In particular, as the closed loop dynamics of a controlled Lagrangian system is itself Lagrangian, it is natural to adopt a variational discretization that exhibits good long-time numerical stability. In the present paper this formalism is developed for the satellite-rotor system. This study is motivated by the importance of structure-preserving algorithms for numerical simulation of the controlled rotating rigid body.

The objective is to spin the rotor in order to stabilize, in the orbital sense, steady-state rotations of the spacecraft about its intermediate inertia axis. Angular momentum conservation is vital for this controller to work, and thus should be incorporated in the discrete model. The configuration space for this problem is the direct product of the rotation groups $\text{SO}(3)$ and $\text{SO}(2)$. In order to obtain the discrete model, it is natural to utilize the Moser–Veselov approach, as it offers an excellent balance of simplicity and quality when capturing the essential features of the rigid body dynamics. We then perform the $\text{SO}(3)$ symmetry reduction and stabilize the equilibria of the reduced system that correspond to the steady-state rotations of interest.

We remark that our methods allow us to simulate and exhibit accurately long time periodic trajectories of the system on the momentum sphere. Such simulations are not possible with conventional integrators.

The paper reveals some unexpected features of the discrete model. While the sets of relative equilibria of the continuous-time system and its discretization are identical, the stability of equilibria of the discrete model is somewhat different from the stability of the corresponding equilibria of the continuous-time system. The stability of the discrete system is sufficient for the stability of its continuous-time counterpart, but not the other way around.

The paper is organized as follows: In Section II we review the continuous-time spacecraft model. The discrete spacecraft model, the discrete matching procedure, the relevant stability analysis, and simulations are presented in Sections III and IV.

II. A SPACECRAFT WITH A SYMMETRIC ROTOR

In this section we overview the rotational dynamics of a spacecraft with a rotor. In particular, we use the matrix representation of angular velocity and momentum, which is
a step towards the transition from the continuous-time to discrete model.

Following Krishnaprasad [24] and Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez [4], we consider a rigid body with a rotor aligned along the third axis of the body as shown in figure 1. The rotor spins under the influence of a control torque \( u \). The configuration space is \( Q = SO(3) \times \mathbb{R}^2 \), with the first and second factors being the configuration spaces of the spacecraft and rotor, respectively. Let \( g \in SO(3) \) be the attitude of the system, \( \Omega = g^{-1} \dot{g} \in \mathfrak{so}(3) \) be the angular velocity of the system measured against a body frame, \( \eta \in \mathbb{R}^3 \) be the angular velocity of the rotor relative to the spacecraft, and \( \mathbb{I}, \mathbb{J} : \mathfrak{so}(3) \rightarrow \mathfrak{so}^*(3) \) be the inertia operators of the spacecraft and the rotor, respectively. In the body frame

\[
\mathbb{I} \dot{\xi} = A \dot{\xi} + \xi A, \quad \mathbb{J} \dot{\xi} = B \dot{\xi} + \xi B, \tag{1}
\]

where \( A = \text{diag}\{A_1, A_2, A_3\} \) and \( B = \text{diag}\{B_1, B_1, B_3\} \). The principle moments of inertia of the spacecraft and rotor are \( I_1 = A_2 + A_3, \ I_2 = A_3 + A_1, I_3 = A_1 + A_2 \) and \( J_1 = B_1 + B_3, J_2 = B_3 + B_1, J_3 = 2B_1 \), respectively. It is assumed that \( A_3 > A_2 > A_1 \).

The Lagrangian equals the total kinetic energy of the system and reads

\[
L = \frac{1}{2} \langle (\mathbb{I} + \mathbb{J}) \Omega, \Omega \rangle + \langle \mathbb{J} \Omega, C \rangle \eta + \frac{1}{2} \langle \mathbb{J} C, C \rangle \eta^2, \tag{2}
\]

where

\[
C = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}. \tag{3}
\]

The dynamics of the spacecraft-rotor system is

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\Omega}} = \text{ad}^*_\Omega \frac{\partial L}{\partial \Omega}, \quad \frac{d}{dt} \frac{\partial L}{\partial \eta} = u, \tag{4}
\]

where \( u \) is the control torque. Recall that \( \partial L/\partial \Omega \in \mathfrak{so}^*(3) \) and \( \partial L/\partial \eta \in \mathbb{R} \) are the angular momentum of the system and the momentum conjugate to the rotor’s angular orientation relative to the body, respectively.

The controlled Lagrangian is selected to be

\[
L_{\tau, \sigma, \rho} = \frac{1}{2} \langle (\mathbb{I} + \mathbb{J}_{\tau, \sigma, \rho}) \Omega, \Omega \rangle + \rho(1 + r) \langle \mathbb{J} \Omega, C \rangle \tau + \frac{1}{2} \rho(\mathbb{J} C, C) \eta^2, \tag{5}
\]

where

\[
\mathbb{J}_{\tau, \sigma, \rho} \xi = B_{\tau, \sigma, \rho} \xi + \xi B_{\tau, \sigma, \rho} \text{ and } B_{\tau, \sigma, \rho} = \text{diag}\{B_1(\rho(1 + r)^2 + \sigma^2), B_1(\rho(1 + r)^2 + \sigma^2), B_3 + B_1(1 - \rho(1 + r)^2 - \sigma^2)\}. \text{ See [8] for motivation and details.}
\]

The dynamics associated with Lagrangian (5) is

\[
\frac{d}{dt} \frac{\partial L_{\tau, \sigma, \rho}}{\partial \eta} = \text{ad}^*_\eta \frac{\partial L_{\tau, \sigma, \rho}}{\partial \Omega} - \frac{d}{dt} \frac{\partial L_{\tau, \sigma, \rho}}{\partial \eta} = 0. \tag{6}
\]

As shown in [8], systems (4) and (6) are equivalent if

\[
\sigma = -1/r, \quad \rho = 1/(1 + r),
\]

and the control torque is defined by

\[
u = -\frac{d}{dt}(2B_1 r \Omega_{12}). \tag{7}
\]

Select \( r \) as the independent gain parameter. Rotations about the intermediate inertia axis are stabilized if

\[
r < \frac{I_3 - I_2 - J_2}{J_3} \quad \text{or} \quad r > \frac{I_3}{J_3}. \tag{8}
\]

Using the quantities \( A_i \) and \( B_i \), the stability conditions read

\[
r > \frac{A_1 + A_2}{2B_1}, \tag{9}
\]

or

\[
r < -1 + \frac{A_2 - A_3 + B_1 - B_3}{2B_1}. \tag{10}
\]

These conditions are equivalent to the stability condition

\[
k > 1 - \frac{I_3}{I_2 + J_2}
\]

of [4], [8], and [14].

III. THE MOSER–VESELOV DISCRETIZATION OF THE SATELLITE-ROTOR

Recall that in the discrete setting, the velocity phase space \( TQ \) is replaced with \( Q \times Q \) and the states of the discrete system are written \([q_k, \dot{q}_{k+1}]\). The discrete dynamics is determined by a discrete analogue of Hamilton’s principle. See [27] for a general exposition of discrete mechanics.

For the satellite-rotor system \( Q = SO(3) \times SO(2) \) and thus \( q_k = (g_k, x_k) \), where \( g_k \in SO(3) \) and \( x_k \in SO(2) \). Define the system’s incremental rotations by the formula \( W_{k,k+1} = g_k^{-1} g_{k+1} \in SO(3) \) and the incremental rotations of the rotor relative to the satellite by \( \Delta x_k = x_{k+1} - x_k \in SO(2) \).

Following the approach of Moser and Veselov (see e.g. [31], [29], [25], [15]), we construct the discrete Lagrangian and discrete controlled Lagrangian for the satellite-rotor system by replacing \( \Omega \) with \( \langle W_{k,k+1} - e \rangle/h \) and \( \eta \) with \( \Delta x_k/h \) in formulae (2) and (5), where \( h > 0 \) is the time-step.

The discretization of the first term in (2) thus becomes

\[
\frac{1}{4} \text{Tr} \left[ W^T_{k,k+1}(A + B)W_{k,k+1} + e^T(A + B)e \\
- W^T_{k,k+1}(A + B)e - e^T(A + B)W_{k,k+1} \\
+ W^T_{k,k+1}W_{k,k+1}(A + B) + e^T e(A + B) \\
- W^T_{k,k+1}(A + B) - W_{k,k+1}(A + B) \right].
\]
Neglecting the constant terms, using the properties of trace, and taking into account that \( A \) and \( B \) are symmetric and \( W_{k,k+1} \) is orthogonal, we obtain
\[
\frac{1}{4} \Tr \left[ -2W_{k,k+1}^T (A + B) - 2W_{k,k+1} (A + B) \right] = \frac{1}{4} \Tr \left[ -4W_{k,k+1} (A + B) \right] = -\Tr W_{k,k+1} (A + B).
\]
Similarly, the discretization of the second term of (2) is
\[
\begin{align*}
\left( \frac{1}{2} \Tr (B(W_{k,k+1} - e) + (W_{k,k+1} - e)B^T)C \right) \Delta x_k &= \frac{1}{2} \Tr (W_{k,k+1}^T BC + BW_{k,k+1}^T C - 2BC) \Delta x_k \\
&= \frac{1}{2} \Tr [BCW_{k,k+1}^T + CBW_{k,k+1}^T - 2BC] \Delta x_k \\
&= \frac{1}{2} \Tr BC(W_{k,k+1}^T - e) \Delta x_k,
\end{align*}
\]
where we used the properties of trace along with the fact that \( CB = BC \) for \( B = \text{diag}\{B_1, B_1, B_3\} \) and for \( C \) defined by formula (3). Finally, the discretization of the last term of (2) becomes
\[
\frac{1}{4} \Tr C^T (BC + CB) \Delta x_k^2 = \frac{1}{4} \Tr [C^T BC + C^T CB] \Delta x_k^2 = \frac{1}{4} \Tr CBC^T \Delta x_k^2.
\]
Summarizing, we arrive at the following statement:

**Theorem 3.1:** The discrete Lagrangian for the satellite-rotor system is given by the formula
\[
L_d(W_{k,k+1}, \Delta x_k) = \frac{1}{4} \left[ -\Tr(W_{k,k+1}(A + B)) \right. \\
+ \Tr (BC(W_{k,k+1} - e)) \Delta x_k + \frac{1}{2} \Tr (CBC^T) \Delta x_k^2 \bigg],
\]
where \( A \) and \( B \) are diagonal matrices that define the inertia operators \( \mathbb{I} \) and \( \mathbb{J} \) as in (1), and \( e \) is the identity matrix.

Note also that, according to the properties of trace, the first term in (11) can be rewritten as
\[
-\Tr ((A + B)W_{k,k+1}) \equiv -\Tr ((A + B)W_{k,k+1}^T) \equiv \Tr (W_{k,k+1}^T (A + B)).
\]

The satellite-rotor system can be interpreted as a generalized rigid body on the group \( SO(3) \times SO(2) \), and thus the controlled dynamics of the discrete satellite-rotor is given by the equations
\[
\begin{align*}
R^*_{W_{k,k+1}+1} D^1 L_d(W_{k,k+1}, \Delta x_k) &= \frac{1}{h} \left[ -\Tr(W_{k,k+1}(A + B)) \right. \\
- \Tr (BC(W_{k,k+1} - e)) \Delta x_k + \frac{1}{2} \Tr (CBC^T) \Delta x_k^2 \bigg], \quad (11)
\end{align*}
\]
where \( \Delta x_k \) is the Lagrange multiplier. See [25] for the derivation of the dynamics of a generalized discrete rigid body. Here and elsewhere, \( L^*_d \) and \( R^*_d \) are the duals of the derivatives of the left and right translations, \( Ad^* \) is the adjoint action of a Lie group on its Lie algebra, and \( Ad^* \) is its dual. Eliminating the Lagrange multiplier, we obtain
\[
\begin{align*}
M_{k+1} - W_{k,k+1}^T M_k W_{k-1,k} &= 0, \quad (14) \\
D_2 L_d(W_{k,k+1}, \Delta x_k) - D_2 L_d(W_{k-1,k}, \Delta x_{k-1}) &= u_k, \quad (15)
\end{align*}
\]
where
\[
M_k = R^*_{W_{k-1,k}} D^1 L_d(W_{k-1,k}, \Delta x_{k-1}) - (R^*_{W_{k-1,k}} D^1 L_d(W_{k-1,k}, \Delta x_{k-1}))^T \quad (16)
\]
is system’s discrete angular momentum. Computing the derivative of the discrete Lagrangian with respect to \( W_{k,k+1} \) and taking into account that \( R^*_d \alpha = \alpha W^T \) for an orthogonal group, we obtain
\[
M_k = W_{k,k+1}(A + B) - (A + B)W_{k-1,k}^T \\
+ (W_{k-1,k} BC + BCW_{k-1,k}^T) \Delta x_{k-1}.
\]
Thus, equation (14) reads
\[
M_{k+1} = (A + B)W_{k,k+1} - W_{k-1,k}(A + B) \\
+ (BCW_{k-1,k} + W_{k-1,k} BC) \Delta x_{k-1}.
\]
The reduced discrete controlled Lagrangian is computed just like the discrete Lagrangian and reads
\[
L^d_{\tau,\rho}(W_{k,k+1}, \Delta x_k) = \frac{1}{2} \left[ -\Tr((W_{k,k+1}(A + B) + \rho(1 + r) Tr(BC(W_{k,k+1} - e)) \Delta x_k + \frac{1}{2} \rho Tr(BC^T) \Delta x_k^2 \right], \quad (17)
\]
The corresponding discrete Euler–Poincaré equations are
\[
\begin{align*}
M^\tau \dot{W}_{k,k+1}^T - W_{k-1,k}^T M^\tau W_{k-1,k} &= 0, \quad (18) \\
D_2 L^d_{\tau,\rho}(W_{k,k+1}, \Delta x_k) - D_2 L^d_{\tau,\rho}(W_{k-1,k}, \Delta x_{k-1}) &= 0, \quad (19)
\end{align*}
\]
where
\[
\begin{align*}
M^\tau &= R^*_{W_{k-1,k}} D^1 L^d_{\tau,\rho}(W_{k-1,k}, \Delta x_{k-1}) - (R^*_{W_{k-1,k}} D^1 L^d_{\tau,\rho}(W_{k-1,k}, \Delta x_{k-1}))^T \\
&= W_{k-1,k}(A + B) - (A + B)W_{k-1,k}^T \\
&+ (W_{k-1,k} BC + BCW_{k-1,k}^T) \Delta x_{k-1}.
\end{align*}
\]
As in the continuous-time case, equation (19) is equivalent to the controlled momentum conservation law
\[
D_2 L^d_{\tau,\rho}(W_{k,k+1}, \Delta x_k) = c = \text{const},
\]
and, just like in the continuous-time case, one selects \( c = 0 \) for studying the discrete analogues of rotations of the satellite about its intermediate axis of inertia. With the condition \( c = 0 \) in mind, the controlled momentum conservation law becomes
\[
\Tr (BC^T) \Delta x_k + (1 + r) \Tr (BC(W_{k,k+1} - e)) = 0.
\]
It is straightforward to check that equations (15) and (19) are equivalent if and only if the discrete control law is
\[
\begin{align*}
u_k &= -\frac{1}{h} \Tr BC \left[ (W_{k,k+1} - e) - (W_{k-1,k} - e) \right] \\
&= -\frac{1}{h} \Tr BC \left[ W_{k,k+1} - W_{k-1,k} \right] \quad (20)
\end{align*}
\]
Indeed, expanding the left-hand sides of equations (15) and (19), we have
\[
\frac{1}{\hbar} \left[ \text{Tr}(CBC^T) \Delta x_{k-1} + \text{Tr}(BC(W_{k-1,k}^T - e)) 
- \text{Tr}(CBC^T) \Delta x_k - \text{Tr}(BC(W_{k,k+1}^T - e)) \right] + u_k = 0,
\]
\[
\frac{1}{\hbar} \left[ \text{Tr}(CBC^T) \Delta x_{k-1} + (1 + r) \text{Tr}(BC(W_{k-1,k}^T - e)) 
- \text{Tr}(CBC^T) \Delta x_k - (1 + r) \text{Tr}(BC(W_{k,k+1}^T - e)) \right] = 0.
\]
The last two equations are equivalent if and only if the discrete control law is given by (20).

Asking that (14) and (18) are equivalent requires that the matching conditions
\[
\sigma = \frac{1}{p}, \quad \rho = \frac{1}{1 + r} \quad (21)
\]
hold. Observe that these matching conditions are identical to the matching conditions of the continuous-time model, and that the discrete control input (20) becomes the control input of the continuous-time model \( \tilde{\sigma} \) after taking the limit \( h \to 0 \).

The dynamics reduced to the zero level of the controlled conservation law\(^1\) reads
\[
M_{k+1} = (A + B)W_{k-1,k} - W_{k-1,k}^T(A + B)
- (BCW_{k-1,k} + W_{k-1,k}BC)
\times \frac{(1 + r) \text{Tr}[BC(W_{k-1,k}^T - e)]}{\text{Tr}[CBC^T]},
\]
where
\[
M_k = W_{k-1,k}(A + B) - (A + B)W_{k-1,k}^T
- (W_{k-1,k}BC + BCW_{k-1,k}^T)
\times \frac{(1 + r) \text{Tr}[BC(W_{k-1,k}^T - e)]}{\text{Tr}[CBC^T]}.
\]
This dynamics can be also written as given by the following implicit map on the group SO(3):
\[
W_{k,k+1}(A + B) - (A + B)W_{k,k+1}^T
- (W_{k,k+1}BC + BCW_{k,k+1}^T)
\times \frac{(1 + r) \text{Tr}[BC(W_{k,k+1}^T - e)]}{\text{Tr}[CBC^T]}
= (A + B)W_{k-1,k} - W_{k-1,k}^T(A + B)
- (BCW_{k-1,k} + W_{k-1,k}BC)
\times \frac{(1 + r) \text{Tr}[BC(W_{k-1,k}^T - e)]}{\text{Tr}[CBC^T]}.
\]

Next, consider relative equilibria that correspond to the rotations of the satellite about the intermediate inertia axis.

\(^1\)Selecting this zero level follows the continuous-time theory and simplifies the exposition without generality loss.

These relative equilibria are represented by the incremental rotations
\[
W_e = \begin{pmatrix} a & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & a \end{pmatrix}.
\]

Since \( W_e \) is an orthogonal matrix, \( a \) and \( b \) should satisfy the condition
\[
a^2 + b^2 = 1.
\]

The corresponding discrete momentum, in the vector form, is
\[
M_e = (0, b(A_1 + B_1 + A_3 + B_3), 0).
\]

a) Linearized Dynamics: In order to linearize the discrete dynamics, we utilize natural charts, see [25] for details. That is, we use local coordinates \( W_e \exp \xi \) near the relative equilibrium (23). Here \( \xi \in \mathfrak{so}(3) \). We write
\[
W_{k,k+1} = W_e \exp \xi_{k,k+1}.
\]

For linearization, it is sufficient to keep only linear terms in (28), so we replace \( W_{k,k+1} \) with
\[
W_e(e + \xi_{k,k+1})
\]
in the reduced dynamics (24).

Writing elements of \( \mathfrak{so}(3) \)
\[
\xi = \begin{pmatrix} 0 & -\xi^2 & \xi^3 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix}
\]
as three-dimensional vectors
\[
\xi = (\xi^1, \xi^2, \xi^3)
\]
and linearizing (24) at \( W_e \), we obtain
\[
A^T \xi_{k,k+1} = A^r \xi_{k-1,k},
\]
where the non-zero components of \( A^T \) and \( A^r \) are
\[
A^T_{11} = A_3 + aA_2 + \frac{1}{2}(r + 1)b^2B_1,
A^T_{13} = bA_2 - \frac{1}{2}(r + 1)(a + 1)bB_1,
A^T_{23} = a(A_1 + A_3),
A^T_{31} = -bA_2 + \frac{1}{2}(r + 1)(a + 1)bB_1,
A^T_{33} = A_1 + aA_2 - \frac{1}{2}(r + 1)(a + 1)^2B_1,
A^r_{11} = A_2 + aA_3 - \frac{1}{2}(r + 1)b^2B_1,
A^r_{13} = bA_3 + \frac{1}{2}(r + 1)(a + 1)bB_1,
A^r_{23} = A_1 + A_3 - \frac{1}{2}(r + 1)(a + 1)^2B_1,
\]
and where \( A_i = A_i + B_i, \ i = 1, 2, 3 \). Straightforward calculation shows that \( \det A^r = \det A^T \) as long as the orthogonality condition (26) is satisfied.
b) Linear Stability: The linearized dynamics is given by the operator

\[ A_d = (A')^{-1} A'. \]  

(29)

The operator \( A_d \) is volume-preserving. One of the eigenvalues of \( A_d \) is always 1 as the family of relative equilibria of the satellite-rotor is one-dimensional. The remaining two eigenvalues belong to the unit circle if

\[ | \text{Tr} \, A_d | < 2. \]  

(30)

Straightforward calculations show that the stability condition (30) is equivalent to the inequality

\[
- A_2 + A_3 + a B_1 + B_3 + r(a + 1) B_1 \\
\times [ A_1 + A_2 + (1 - a) B_1 - r(1 + a) B_1 ] < 0.
\]  

(31)

Since each of the quantities \( A_1, A_3, B_1, B_3 \) is positive, the latter formula implies that either

\[ r > \frac{A_1 + A_2 + B_1 - a B_1}{(1 + a) B_1}, \]  

(32)

or

\[ r < -1 + \frac{A_2 - A_3 + B_1 - B_3}{(1 + a) B_1}. \]  

(33)

For sufficiently small time steps, relative equilibria are represented by incremental rotations (25) through an acute angle so that \( 0 \leq a < 1 \), which implies \( 1 \leq 1 + a < 2 \). Therefore, (32) implies the continuous-time stability condition (9).

Assume that

\[ A_2 - A_3 + B_1 - B_3 < 0. \]  

(34)

Then (33) implies the continuous-time stability condition (10).

We thus have the following result:

**Theorem 3.2:** Assume (34) holds. Then conditions (32) and (33) for linear stability of the relative equilibrium (25) of the discrete controlled satellite-rotor imply the stability condition (9) and (10) of the corresponding relative equilibrium of the original continuous-time controlled satellite-rotor.

Thus, if the Moser–Veselov algorithm is used for discretization, there exist values of the gain parameter \( r \) such that the continuous-time model is stabilized but the discrete model is not. For these \( r \) values, the discrete model of the satellite-rotor cannot be used to simulate the system.

**IV. SIMULATIONS**

We validate the analysis of section III by simulations of the dynamics of discrete satellite-rotor with \( A_1 = 1 \text{ kg} \cdot \text{m}^2 \), \( A_2 = 2 \text{ kg} \cdot \text{m}^2 \), \( A_3 = 5 \text{ kg} \cdot \text{m}^2 \), \( B_1 = 2 \text{ kg} \cdot \text{m}^2 \), \( B_3 = .3 \text{ kg} \cdot \text{m}^2 \), and for the incremental rotation (25) with \( a = .99 \) and \( b = \sqrt{1 - a^2} \). The corresponding momentum is given by formula (27).

For simulating the update maps, we utilize the unit quaternion representation of the rotation group SO(3). This approach leads to an efficient code and complements the model of [26], [32], where a discrete model of a rigid body using quaternions on the full phase space, i.e., on \( \text{SO}(3) \times \text{SO}(3) \), was introduced.

Figure 2 obtained by iterating the Moser–Veselov update map, shows the n-dimensional momentum sphere an illustrates instability of the said rotation for the uncontrolled satellite.

![Fig. 2. Momentum dynamics for the uncontrolled rigid body](image1.png)

Finally, we set \( r = -10 \), so that condition (34) is satisfied, and iterate the update map (24) for the controlled satellite. The resulting momentum dynamics, shown in figure 3 confirms that stabilization of the incremental rotation (25) takes place.

![Fig. 3. Momentum dynamics for the satellite-rotor, \( r = -10 \)](image2.png)

These simulations demonstrate that the algorithm does a very good job computing periodic trajectories. In particular, the loops that surround the north and south poles of the momentum sphere in figure 4 have very long periods, and the algorithm captures these trajectories well.

![Fig. 4. Momentum dynamics for the satellite-rotor, \( r = -8.76 \)](image3.png)
V. CONCLUSIONS

In this paper we introduced the discrete satellite-rotor system and obtained a discrete stabilizing feedback control law for this system. This development revealed in particular a distinction between the stability conditions for the continuous-time system and its discretization. The origins of this difference can be traced to the discrete version of Routh reduction. In a future publication we will address this issue in detail and consider variations of the discretization process and its relationship to stability of the continuous system.

The method in this paper is related to other discrete methods in control that have a long history; recent papers that use discrete mechanics in the context of optimal control and celestial navigation are [20], [23], and [30]. We intend to use the theory developed here to address stabilization and control problems of other coupled rigid body systems including the rigid body with multiple rotors.

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