Matching and Stabilization of Linear Mechanical Systems

Dmitry V. Zenkov
Department of Mathematics
North Carolina State University
Raleigh, NC 27695
USA

Abstract

We consider linear controlled mechanical systems and show that controllability enables one to use the method of controlled Lagrangians for feedback stabilization of equilibria.

1 Introduction

The method of *controlled Lagrangians* for holonomic mechanical systems originated in Bloch, Marsden, and Sánchez de Alvarez [7] and Bloch, Leonard, and Marsden [3], and was then developed in Auckly, Kapitanski, and White [1], Bloch, Leonard, and Marsden [4, 5, 6], Bloch, Chang, Leonard, and Marsden [2], and Hamberg [10, 11]. In [15, 16], Zenkov, Bloch, Leonard, and Marsden extended this method to a class of nonholonomic systems.

This method is based on the requirement that the closed-loop dynamics is derived from the new, *controlled* Lagrangian. This Lagrangian is typically represented as the difference in modified kinetic and potential energy of the original system. The new terms that appear in the equations of motion define the control inputs. The closed-loop dynamics obtained this way has a natural conservation law—the energy associated with the controlled Lagrangian.

A controlled system is called *underactuated* if the control forces are allowed in certain directions only. Asking that the controlled dynamics remains Lagrangian and that the control forces appear in the desired directions only imposes certain *matching conditions* on the original and controlled Lagrangians. The matching conditions are represented by an overdetermined system of partial differential equations. Some results on the compatibility of this system and on how one solves these equations can be found in Auckly, Kapitanski, and White [1] and Chang [8].

For stabilization of an equilibrium of the original system, one needs to be able to construct a controlled Lagrangian whose kinetic and potential energies are *positive-definite* at this equilibrium. It is unclear whether compatibility of the matching conditions is sufficient for existence of such a Lagrangian. In particular, if one uses the method of characteristics for solving the matching conditions (see Hamberg [10]), the initial conditions for the controlled kinetic and potential energies should produce a controlled Lagrangian whose energy is positive-definite and thus have to satisfy certain restrictions.

In the present paper we address the problem of existence of a stabilizing controlled Lagrangian for *linear mechanical systems.* The matching conditions in this situation are represented by a matrix equation for the kinetic and potential energy forms. We prove that controllability of the original system implies the existence of solutions of this equation in the class of *symmetric positive-definite matrices.* This result can be useful in obtaining the proper initial values for the nonlinear matching conditions.

The paper is organized as follows: In section 2 we give a brief overview of the method of controlled Lagrangians and the nature of the matching conditions. In section 3 we discuss linear controlled mechanical systems and introduce the Hamiltonian representation for the controlled dynamics. The main results are described in section 4; we first introduce the feedback control inputs that place the eigenvalues of the system on the imaginary axis, and then conclude that the controlled dynamics has a positive-definite quadratic conservation law and construct a positive-definite controlled Lagrangian by modifying this conserved quantity.

2 An Overview of the Method of Controlled Lagrangians

Consider a controlled mechanical system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} + b_i^k(q,\dot{q})u_k, \qquad i=1,\ldots,n, \qquad k=1,\ldots,m, \tag{2.1}$$

where u_k are the control inputs. Throughout the paper, all indices range from 1 to n unless otherwise stated, and a summation over repeated indices is understood. The Lagrangian has the form of kinetic minus potential energy:

$$L = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j - U(q).$$

We assume that this system is *underactuated, i.e.,* the number of control inputs, m, is strictly smaller than the number of degrees of freedom of (2.1), n. Let q_0 be an unstable equilibrium of (2.1). The method of controlled Lagrangians suggests the following strategy for stabilization of this equilibrium:

1. Introduce a new function $\widetilde{L} = \frac{1}{2}\widetilde{g}_{ij}(q)\dot{q}^i\dot{q}^j - \widetilde{U}(q)$ called the *controlled Lagrangian.*

2. Require that the original controlled dynamics (2.1) is equivalent to the *uncontrolled* dynamics associated with $\widetilde{L}$,

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{q}^i} = \frac{\partial \widetilde{L}}{\partial q^i}, \qquad i=1,\ldots,n. \tag{2.2}$$

   This equivalence determines the control inputs (see theorem 2.1 below).

3. Adjusting $\widetilde{L}$ if necessary, make the equilibrium q_0 of (2.2) neutrally stable.

4. If asymptotic stabilization is desired, add dissipation emulation terms to the control inputs u_k.

The conditions for equivalence of systems (2.1) and (2.2) are called the *matching conditions.* They are specified in the following theorem (see Hamberg [10] for details).

Theorem 2.1. *Equations (2.1) and (2.2) are equivalent if and only if the following matching conditions hold:*

$$c^i_\beta g_{ij}(\Gamma^j_{ab} - \widetilde{\Gamma}^j_{ab}) = 0, \qquad c^i_\beta \left(\frac{\partial U}{\partial q^i} - g_{ij}\widetilde{g}^{jk}\frac{\partial \widetilde{U}}{\partial q^k} \right) = 0, \qquad \beta = 1, \ldots, n-m. \tag{2.3}$$

The control inputs in this case can be obtained from the equations

$$b^k_i(q,\dot{q})u_k = \frac{\partial U}{\partial q^i} - g_{ij}\widetilde{g}^{ja}\frac{\partial \widetilde{U}}{\partial q^a} + g_{ij}(\Gamma^j_{ab} - \widetilde{\Gamma}^j_{ab})\dot{q}^a\dot{q}^b, \qquad k = 1, \ldots, m.$$

In (2.3), Γ^j_{ab} and $\widetilde{\Gamma}^j_{ab}$ represent the Christoffel symbols for the metrics g_{ij} and $\widetilde{g}_{ij}$, respectively, and the coefficients c^i_β are determined by

$$b^j_i c^i_\beta = 0, \qquad j = 1, \ldots, m, \qquad \beta = 1, \ldots, n-m.$$

3 Linear Lagrangian Systems

In this section we introduce linear Lagrangian systems along with some useful coordinate transformations and write out the matching conditions for these systems.

Linear Controlled Lagrangian Systems. Consider a quadratic Lagrangian

$$L = \frac{1}{2}(g_{ij}\dot{q}^i\dot{q}^j - a_{ij}q^iq^j), \qquad i,j = 1, \ldots, n, \tag{3.4}$$

and the linear controlled mechanical system associated with (3.4):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} + b^k_i u_k, \tag{3.5}$$

where $(u_1, \ldots, u_m)$ are the control inputs. Here and below, g_{ij} is a positive-definite constant matrix, and a_{ij} and b^k_i are constant matrices. In the absence of controls the origin is an unstable equilibrium of (3.5). We assume that system (3.5) is controllable, that is, after rewriting (3.5) as a system of first order ordinary differential equations

$$\dot{z} = Az + Bu, \tag{3.6}$$

the *controllability rank condition*

$$\operatorname{rank}\left(B, AB, A^2B, \ldots, A^{2n-1}B\right) = 2n \tag{3.7}$$

is satisfied. For an underactuated system, the number of independent control inputs equals $\operatorname{rank} b^k_i$. We thus assume $\operatorname{rank} b^k_i = m$.

The Matching Conditions. To stabilize the equilibrium $q = 0$ of (3.5), we introduce the *quadratic* controlled Lagrangian

$$\widetilde{L} = \frac{1}{2}(\widetilde{g}_{ij}\dot{q}^i\dot{q}^j - \widetilde{a}_{ij}q^iq^j).$$

The matching conditions (2.3) reduce to the following equation for matrices $\widetilde{g}^{jk}$ and $\widetilde{a}^{jk}$:

$$c^i_\beta(g_{ij}\widetilde{g}^{jk} - a_{ij}\widetilde{a}^{jk}) = 0, \qquad \beta = 1,\dots,n-m, \qquad k = 1,\dots,n.$$

The solutions of this equation should belong to the class of symmetric positive-definite matrices. Below we show that such solutions exist if the controllability condition (3.7) holds.

Let $\Lambda_1,\dots,\Lambda_n$ be the roots of the equation $\det(a_{ij} - \lambda g_{ij}) = 0$. There always exists a linear substitution $q = Tx$ that transforms the matrices g_{ij} and a_{ij} into the unit matrix I and the diagonal matrix $\Lambda = \operatorname{diag}(\Lambda_1,\dots,\Lambda_n)$, respectively.

The Hamiltonian Representation. One of the ways to represent (3.5) as (3.6) is to rewrite the original system as a Hamiltonian one (we refer the reader to [9] and [14] for details on Lagrangian and Hamiltonian representations of mechanical control systems). Introduce the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n}(y_i^2 + \Lambda_i x_i^2).$$

As usual, $y_i = \partial_{\dot{x}_i}L = \dot{x}_i$ are the conjugate momenta. The equations of motion written in the variables (x, y) become

$$\dot{x} = y, \qquad \dot{y} = -\Lambda x + Bu. \tag{3.8}$$

These have the form of (3.6) if we put

$$z = (x, y), \qquad A = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

4 Existence of Stabilizing Quadratic Controlled Lagrangians

In this section we prove that the origin equilibrium of a linear controllable mechanical system can be stabilized by the method of controlled Lagrangians.

The Feedback Choice. We choose the feedback control u in (3.8) such that all eigenvalues of (3.8) are distinct pure imaginary conjugate pairs. The controllability assumption enables us to do that. Moreover, we can choose u that depends linearly on the configuration coordinates x only and does not depend on the momenta y.

Lemma 4.1. *There exists an $m \times n$ matrix K such that the linear positional feedback control $u = Kx$ places the eigenvalues of (3.8) on the imaginary axis.*

Proof. If $u = Kx$, equations (3.8) are equivalent to

$$\ddot{x} = (-\Lambda + bK)x,$$

and therefore the characteristic polynomial of (3.8) with $u = Kx$ may be written as

$$p(\lambda) = \det(\lambda^2 I - (-\Lambda + bK)).$$

One can check that the controllability condition (3.7) is equivalent to

$$\text{rank}(b, \Lambda b, \Lambda^2 b, \ldots, \Lambda^{n-1} b) = n.$$

Therefore the eigenvalues of the matrix $-\Lambda + bK$ may be assigned any values by an appropriate choice of the matrix K. In our case, making the eigenvalues of $-\Lambda + bK$ distinct negative real numbers places the eigenvalues of (3.8) on the imaginary axis. □

The Structure of the Controlled System. With this choice of controls, (3.8) becomes the system of $2n$ linear ordinary differential equations

$$\dot{z} = \mathcal{A}z. \tag{4.9}$$

The matrix of this system is

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ F & 0 \end{pmatrix}, \qquad \text{where} \qquad F = -\Lambda + bK.$$

Since all eigenvalues of our controlled system are distinct pure imaginary pairs, this system has a positive-definite quadratic integral

$$f(x, y) = \frac{1}{2} z^T \mathcal{B} z, \tag{4.10}$$

where $\mathcal{B}$ is a symmetric $2n \times 2n$ matrix, which we write as

$$\mathcal{B} = \begin{pmatrix} C & E \\ E^T & D \end{pmatrix}, \qquad C = C^T, \qquad D = D^T.$$

The origin is a stable equilibrium of (4.9). Observe that C and D are positive-definite $n \times n$ matrices.

According to Kozlov [12, 13], a *non-degenerate* system of $2n$ linear differential equations (4.9) that has a quadratic integral (4.10) is Hamiltonian: The matrix

$$\Omega = \mathcal{B}\mathcal{A}^{-1} \tag{4.11}$$

is non-degenerate and skew-symmetric and therefore defines a *symplectic structure*, which allows one to represent equations (4.9) as

$$\dot{z} = \Omega^{-1} df(z).$$

In the next paragraph we will construct the controlled Lagrangian from the Hamiltonian (4.10).

The Controlled Lagrangian. We now prove that the equilibrium $q = 0$ of (3.5) can be stabilized by a suitable choice of a quadratic controlled Lagrangian.

Theorem 4.1. *If the system*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i} + b_i^k u_k \tag{4.12}$$

is controllable, then there exists a controlled Lagrangian $\widetilde{L} = (\dot{x}^T \widetilde{G} \dot{x} - x^T \widetilde{A} x)/2$ *such that the system*

$$\frac{d}{dt}\frac{\partial \widetilde{L}}{\partial \dot{x}^i} = \frac{\partial \widetilde{L}}{\partial x^i}$$

is equivalent to (4.12) and the energy $\widetilde{E} = (\dot{x}^T \widetilde{G} \dot{x} + x^T \widetilde{A} x)/2$ *associated with* $\widetilde{L}$ *is positive-definite. The control law is given by* $u = Kx$*, where* K *is determined by the equation*

$$bK = \Lambda - \widetilde{G}^{-1}\widetilde{A}. \tag{4.13}$$

Proof. According to lemma 4.1, the controllability of (4.12) implies the existence of a positive-definite Hamiltonian (4.10) and a symplectic structure (4.11). We now write out explicitly the condition for (4.10) to be an integral of (4.9). Differentiating (4.10) along the flow (4.9), we obtain

$$z^T \begin{pmatrix} EF & C \\ DF & E^T \end{pmatrix} z \equiv 0.$$

The matrix

$$\begin{pmatrix} EF & C \\ DF & E^T \end{pmatrix}$$

is therefore skew-symmetric, which implies

$$C = -DF, \qquad E = -E^T, \qquad (EF)^T = -EF. \tag{4.14}$$

Next, we set $E = 0$ and obtain a new positive-definite quadratic integral

$$\widetilde{H} = \frac{1}{2}(y^T D y + x^T C x)$$

of (4.9). We then define the controlled Lagrangian $\widetilde{L}$ by setting $\widetilde{G} = D$ and $\widetilde{A} = C$:

$$\widetilde{L} = \frac{1}{2}(\dot{x}^T D \dot{x} - x^T C x). \tag{4.15}$$

Consider now a Lagrangian system associated with (4.15):

$$D\ddot{x} = -Cx.$$

This system may be rewritten as

$$\ddot{x} = -D^{-1}Cx,$$

which according to (4.14) becomes

$$\ddot{x} = Fx.$$

The last system is equivalent to the original system (4.12) with the control $u = Kx$. The equations $F = -\Lambda + bK$ and (4.14) imply (4.13). □

Acknowledgment: I would like to thank professors A.M. Bloch, J.E. Marsden, and L.K. Norris for helpful discussions. Research partially supported by NSF grant DMS-9803181, AFOSR grant F49620-96-1-0100, a University of Michigan Rackham Fellowship, and an NSF group infrastructure grant at the University of Michigan.

References

[1] Auckly, D., L. Kapitanski, & W. White, Control of Nonlinear Underactuated Systems, *Commun. Pure Appl. Math.* **53**, 2000, 354–369.

[2] Bloch, A.M., D-E. Chang, N.E. Leonard, & J.E. Marsden, Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping, *IEEE Trans. Autom. Control* **46**, 2001, 1556–1571.

[3] Bloch, A.M., N.E. Leonard, & J.E. Marsden, Stabilization of Mechanical Systems Using Controlled Lagrangians, *Proc. CDC* **36**, 1997, 2356–2361.

[4] Bloch, A.M., N.E. Leonard, & J.E. Marsden, Matching and Stabilization by the Method of Controlled Lagrangians, *Proc. CDC* **37**, 1998, 1446–1451.

[5] Bloch, A.M., N.E. Leonard, & J.E. Marsden, Potential Shaping and the Method of Controlled Lagrangians, *Proc. CDC* **38**, 1999, 1652–1657.

[6] Bloch, A.M., N.E. Leonard, & J.E. Marsden, Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem, *IEEE Trans. Autom. Control* **45**, 2000, 2253–2270.

[7] Bloch, A.M., J.E. Marsden, & Sánchez de Alvarez, Stabilization of Relative Equilibria of Mechanical Systems with Symmetry, in *Current and Future Directions in Applied Mathematics*, (M. Alber, B. Hu, and J. Rosenthal, eds.), Birkhäuser, 1997, 43–64.

[8] Chang, D-E., The Extended λ-method for Controlled Lagrangians, *preprint,* 2002.

[9] Chang, D-E., A.M. Bloch, N.E. Leonard, J.E. Marsden, & C.A. Woolsey, The Equivalence of Controlled Lagrangian and Controlled Hamiltonian Systems, to appear in *ESAIM: Control, Optimization and Calculus of Variations,* 2002.

[10] Hamberg, J., General Matching Conditions in the Theory of Controlled Lagrangians, *Proc. CDC* **38**, 1999, 2519–2523.

[11] Hamberg, J., Controlled Lagrangians, Symmetries and Conditions for Strong Matching, in *Lagrangian and Hamiltonian Methods for Non-linear Control,* Elsevier, 2000.

[12] Kozlov, V.V., Linear Systems with a Quadratic Integral, *J. Appl. Math. Mech.* **56**, 1992, 803–809.

[13] Kozlov, V.V., *Symmetries, Topology, and Resonances in Hamiltonian Mechanics,* Springer-Verlag, 1996.

[14] Ortega, R., M.W. Spong, F. Gómez-Estern & G. Blankenstein, Stabilization of Underactuated Mechanical Systems via Interconnection and Damping Assignment, *IEEE Trans. Autom. Control (submitted)*, 2001.

[15] Zenkov, D.V., A.M. Bloch, N.E. Leonard, & J.E. Marsden, Matching and Stabilization of Low-Dimensional Nonholonomic Systems, *Proc. CDC* **39**, 2000, 1289-1295.

[16] Zenkov, D.V., A.M. Bloch, & J.E. Marsden, Flat Nonholonomic Matching, *Proc. ACC*, 2000, 2812–2817.