

# DYNAMICS OF THE DISCRETE CHAPLYGIN SLEIGH

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**Abstract.** This paper studies the dynamics of the discrete Chaplygin sleigh. Properties such as discrete momentum and measure conservation are explored.

## 1 Introduction

The theory of variational integrators for Lagrangian and Hamiltonian systems originated in [16], [17], and [14]. It was further developed by a number of authors (see e.g. [4], [13], [10], [18], and [12] for a more complete list of references and history). A very important feature of variational integrators is the *discrete momentum preservation*: if the original continuous-time system has symmetry and conserves the momentum map, so does the associated discrete-time mechanical system.

In [6], the theory of variational integrators was adapted to Lagrangian nonholonomic systems. In particular, it is shown in [6] that the discrete-time nonholonomic system conserves the *spatial* momentum in the case of *horizontal symmetry* (see [3] for the definition of horizontal symmetry). However, horizontal symmetry is not typical in nonholonomic mechanics. Apparently, Chaplygin [5] was the first to observe the link between symmetry and conservation of the components of momentum along a *moving* frame; see also [21] and references therein. Therefore, it is natural to ask whether the discrete momentum is preserved by the discrete-time nonholonomic system associated with a momentum-preserving continuous-time system in the case of nonhorizontal symmetry. A closely related property is the existence of an invariant measure. Continuous-time nonholonomic systems generically are not measure-preserving (see [8] and [20] for details).

This paper addresses the momentum and measure preservation properties for the discrete-time *Chaplygin sleigh*. The Chaplygin sleigh is the system introduced and studied by Chaplygin in [5]. This system is a rigid body on a horizontal plane constrained by a blade. The blade limits the velocity of the body-plane contact point to a direction fixed in the body. We assume that the center of mass of the body belongs to the line through the blade. If the center of mass of the body coincides with the contact point, the dynamics conserves the angular and linear momenta of the system *relative to the body frame*. The symmetry in this problem is therefore non-horizontal. If the center of mass is off the contact point, the momentum is no longer conserved. Instead, heteroclinic connections appear between the pairs of equilibria in the momentum plane.

We show that the discrete model of the Chaplygin sleigh demonstrates the same behavior, that is, the discrete momentum relative to the body frame is conserved if the center of mass is at the contact point, and the momentum dynamics consists of equilibria and heteroclinic connections if the center of mass is off the contact point.

The exposition is organized as follows: In Section 2, basic facts about continuous-time and discrete-time nonholonomic dynamics are reviewed. In Section 3, the dynamics of the Chaplygin sleigh is discussed. Section 4 introduces the discrete model of the Chaplygin sleigh and studies its dynamics.

We remark that this paper develops an elementary approach to the discrete dynamics of the Chaplygin sleigh. The general approach based on the use of Lie groups will be the subject of a future publication.

## 2 An Overview of Nonholonomic Dynamics

In this section we briefly discuss the main concepts of nonholonomic dynamics. For a complete exposition see [2] and [3].

### 2.1 The Euler–Lagrange Equations for Nonholonomic Systems

A nonholonomic Lagrangian system is a triple  $(Q, L, \mathcal{D})$ , where  $Q$  is a smooth  $n$ -dimensional manifold called the *configuration space*,  $L : TQ \rightarrow \mathbb{R}$  is a smooth function called the *Lagrangian*, and  $\mathcal{D} \subset TQ$  is a  $m$ -dimensional *constraint distribution*. Recall that a distribution  $\mathcal{D}$  is a collection of linear subspaces  $\mathcal{D}_q \subset T_qQ$ , one for each  $q \in Q$ . Let  $q = (q^1, \dots, q^n)$  be local coordinates on  $Q$ . In the induced coordinates  $(q, \dot{q})$  on the tangent bundle  $TQ$  we write  $L(q, \dot{q})$ . It is assumed that the Lagrangian is *hyperregular*, i.e., the map

$$\frac{\partial L}{\partial \dot{q}} : TQ \rightarrow T^*Q$$

is invertible (see [11]).

A curve  $q(t) \in Q$  is said to *satisfy the constraints* if  $\dot{q}(t) \in \mathcal{D}_{q(t)}$  for all  $t$ . The equations of motion are given by the following Lagrange–d’Alembert principle: *The equations of motion for the system are those determined by*

$$\delta \int_a^b L(q^i, \dot{q}^i) dt = 0, \tag{1}$$

where we choose variations  $\delta q(t)$  of the curve  $q(t)$  that satisfy  $\delta q(a) = \delta q(b) = 0$  and  $\delta q(t) \in \mathcal{D}_{q(t)}$  for each  $t$  where  $a \leq t \leq b$ . This principle is supplemented by the condition that the curve itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations. Taking variations after imposing the constraints results in variational nonholonomic, or vakonomic, mechanics. The latter is appropriate for optimal control problems. See Bloch [2] and Bloch, Krishnaprasad, Marsden, and Murray [3] for a discussion and references.

Assuming that the constraint distribution is specified by a set of  $n-m$  differential forms  $A^j(q)$ ,  $j = 1, \dots, n-m$ , that is,

$$\mathcal{D} = \{\dot{q} \in TQ \mid \langle A^j(q), \dot{q} \rangle = 0, j = 1, \dots, s = n-m\}, \tag{2}$$

equation (1) implies

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \sum_{j=1}^s \lambda_j A^j(q). \tag{3}$$

Equations (3) are called the *Euler–Lagrange equations with multipliers*. Coupled with (2), they give a complete description of the dynamics of the system.

## 2.2 Discrete Mechanical Systems with Nonholonomic Constraints.

Let  $Q$  be a smooth manifold. According to [6], a discrete nonholonomic mechanical system on  $Q$  is characterized by

- (i) a *discrete Lagrangian*  $L_d : Q \times Q \rightarrow \mathbb{R}$ ;
- (ii) an  $(n - s)$ -dimensional distribution  $\mathcal{D}$  on  $TQ$ ;
- (iii) a *discrete constraint manifold*  $\mathcal{D}_d \subset Q \times Q$  which has the same dimension as  $\mathcal{D}$  and satisfies the condition  $(q, q) \in \mathcal{D}_d$  for all  $q \in Q$ .

The dynamics is given by the following *discrete Lagrange-d'Alembert principle* (see [6]):

$$\sum_{k=0}^{N-1} \left( D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) \right) \delta q_k = 0, \quad \delta q_k \in \mathcal{D}_{q_k}, \quad (q_k, q_{k+1}) \in \mathcal{D}_d.$$

Here  $D_1 L_d$  and  $D_2 L_d$  denote the partial derivatives of the discrete Lagrangian with respect to the first and the second inputs, respectively.

The discrete constraint manifold is usually specified by the *discrete constraint functions*

$$\mathcal{F}_j(q_k, q_{k+1}) = 0, \quad j = 1, \dots, s. \quad (4)$$

Let the distribution  $\mathcal{D}$  be given by a set of one-forms  $A^j(q)$  as in (2). The dynamics of a discrete nonholonomic system is represented by sequences  $\{(q_k, q_{k+1})\}$  that satisfy the *discrete Euler-Lagrange equations with multipliers*

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = \sum_{j=1}^s \lambda_j^k A^j(q_k) \quad (5)$$

coupled with equations (4). According to [6], the map  $(q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$  defined by equations (4) and (5) is a local diffeomorphism in a neighborhood of the diagonal of  $Q \times Q$  if the matrix

$$\begin{pmatrix} D_1 D_2 L_d(q, q') & A^1(q) & \dots & A^s(q) \\ D_2 \mathcal{F}_1(q, q') & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ D_2 \mathcal{F}_s(q, q') & 0 & \dots & 0 \end{pmatrix}$$

is invertible for each  $(q, q')$  from this neighborhood.

One way to construct the discrete Lagrangian is to set  $L_d = L \circ \Psi$ , where  $\Psi : Q \times Q \rightarrow TQ$  is the *discretization map*. In this case the discrete constraint manifold  $\mathcal{D}_d$  has to be consistent with the distribution  $\mathcal{D}$ , that is,  $\mathcal{D}_d$  has to be locally defined as  $A^j \circ \Psi = 0$ ,  $j = 1, \dots, s$ .<sup>1</sup> If the configuration space is  $\mathbb{R}^n$ , it is natural to choose

$$\Psi(q_k, q_{k+1}) = \left( \frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h} \right) \in T\mathbb{R}^n, \quad (6)$$

<sup>1</sup>More generally, the discrete constraints are consistent with the continuous ones if the tangent vectors to the curves in  $\mathcal{D}_d \subset Q \times Q$  at the diagonal of  $Q \times Q$  span the distribution  $\mathcal{D} \subset TQ$ .

where  $h \in \mathbb{R}_+$  is the *time step* (see [10] and [6] for details). We emphasize that the discretization map is not unique and hence there are many ways to define the discrete Lagrangian  $L_d$  and the discrete constraint manifold  $\mathcal{D}_d$  for a given nonholonomic system  $(Q, L, \mathcal{D})$ .<sup>2</sup>

### 3 The Chaplygin Sleigh

The Chaplygin sleigh is a rigid body with a blade attached, moving on a horizontal plane. The blade allows no motion orthogonal to its direction. This mechanical system was introduced and studied by Chaplygin [5]. The configuration space of this system is  $SE(2)$ , the group of Euclidean motions of the two-dimensional plane  $\mathbb{R}^2$ . We parameterize this group with coordinates  $(\theta, x, y)$ , where  $\theta$  is the angular orientation of the blade and  $(x, y)$  is the position of the contact point of the blade and the plane.

The Lagrangian equals the kinetic energy of the body, which is a sum of the kinetic energy of the center of mass and the kinetic energy of the rotational mode of the body. We assume here that the center of mass is situated on the line through the blade. Denote the moment of inertia of the body relative to its center of mass, the mass of the body, and the distance from the center of mass to the contact point of the blade and the plane by  $J$ ,  $m$ , and  $a$ , respectively. The absolute position of the center of mass is  $(x + a \cos \theta, y + a \sin \theta)$ , and thus, the speed of the center of mass is

$$\left[ \dot{x}^2 + \dot{y}^2 + a^2 \dot{\theta}^2 + 2a\dot{\theta}(-\dot{x} \sin \theta + \dot{y} \cos \theta) \right]^{1/2}.$$

The Lagrangian becomes

$$L = \frac{1}{2} \left[ (J + ma^2) \dot{\theta}^2 + m \left( \dot{x}^2 + \dot{y}^2 + 2a\dot{\theta}(-\dot{x} \sin \theta + \dot{y} \cos \theta) \right) \right]. \quad (7)$$

The constraint is

$$-\dot{x} \sin \theta + \dot{y} \cos \theta = 0. \quad (8)$$

The dynamics of the Chaplygin sleigh is governed by the Euler–Lagrange equations (3) coupled with the constraint equation (8).

The dynamics of the *balanced* Chaplygin sleigh ( $a = 0$ ) is drastically different from the dynamics of the *unbalanced* sleigh ( $a > 0$ ). For example, the trajectory of the contact point of the balanced sleigh and the plane is generically a circle. Looking ahead to Figure 3, the solid curve shows a typical trajectory of the contact point in the unbalanced case. We remark that the shape of this curve is completely determined by the value of the parameters  $J$ ,  $m$ , and  $a$  (see [15]).

Many important properties of this dynamics can be obtained from the study of the evolution of the *nonholonomic momentum* summarized in the three theorems that follow.

The nonholonomic momentum for the Chaplygin sleigh has two components,

$$p_1 = (J + ma^2) \dot{\theta} \quad \text{and} \quad p_2 = m(\dot{x} \cos \theta + \dot{y} \sin \theta), \quad (9)$$

which are the angular momentum of the body relative to the vertical axis through the contact point and the projection of the linear momentum on the direction of

<sup>2</sup>An alternative approach to the discretization of nonholonomic systems based on a modification of canonical transformations was proposed in [9].

the blade, respectively (see [3] for the definition of the nonholonomic momentum). The momentum dynamics is governed by the equations

$$\dot{p}_1 = -\frac{ap_1p_2}{J+ma^2}, \quad \dot{p}_2 = \frac{map_1^2}{(J+ma^2)^2} \quad (10)$$

(see [2] for the derivation of these equations).

**Theorem 1.** *The momentum dynamics (10) preserves the constrained energy*

$$\frac{1}{2} \left( \frac{p_1^2}{J+ma^2} + \frac{p_2^2}{m} \right). \quad (11)$$

**Theorem 2.** *The nonholonomic momentum is preserved if and only if the center of mass of the body is located at the contact point, i.e.,  $a = 0$ .*

**Theorem 3.** *In the general, unbalanced case, equations (10) have a one-dimensional manifold of equilibria  $p_1 = 0$ . The equilibria  $(0, p_2)$  are stable, and asymptotically stable with respect to  $p_1$ , if  $p_2 > 0$ , and are unstable if  $p_2 < 0$ . Each pair of equilibria  $(0, p_2)$  and  $(0, -p_2)$  is connected by a pair of heteroclinic trajectories.*

**Corollary 1.** *The momentum dynamics is volume-preserving if and only if  $a = 0$ .*

This momentum dynamics is illustrated in Figure 1; the stable and unstable equilibria are represented by the filled and empty dots, respectively. The heteroclinic trajectories are the level curves of the constrained energy (11).

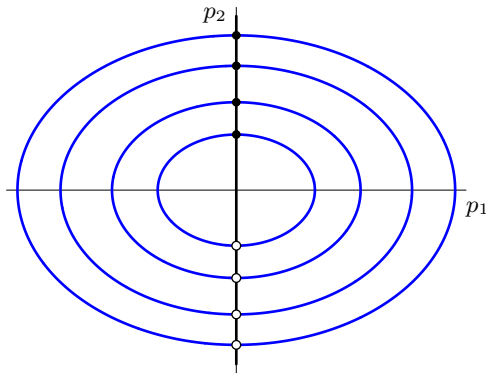


Figure 1: Momentum dynamics of the unbalanced Chaplygin sleigh.

## 4 The Discrete Chaplygin Sleigh

In this Section we derive the equations of motion of the discrete Chaplygin sleigh and analyze the dynamics governed by these equations.

## 4.1 The Discrete Dynamics

Let  $\Delta\theta_k = \theta_{k+1} - \theta_k$ ,  $\Delta x_k = x_{k+1} - x_k$ , and  $\Delta y_k = y_{k+1} - y_k$ . Since  $\mathbb{R}^3$  is a covering space for the group  $SE(2) = \mathbb{R}^3/\mathbb{Z}$ , we use (6) to obtain the discrete Lagrangian

$$L_d = \frac{1}{2h^2} \left[ (J + ma^2)\Delta\theta_k^2 + m \left( \Delta x_k^2 + \Delta y_k^2 + 2a\Delta\theta_k \left( -\Delta x_k \sin \frac{\theta_{k+1} + \theta_k}{2} + \Delta y_k \cos \frac{\theta_{k+1} + \theta_k}{2} \right) \right) \right] \quad (12)$$

and the discrete constraint

$$-\Delta x_k \sin \frac{\theta_{k+1} + \theta_k}{2} + \Delta y_k \cos \frac{\theta_{k+1} + \theta_k}{2} = 0. \quad (13)$$

The factor  $1/h^2$  in (12) does not affect the dynamics and is omitted in the rest of the paper. An equivalent representation of (13) is

$$-(\Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k) \sin \frac{\Delta\theta_k}{2} + (-\Delta x_k \sin \theta_k + \Delta y_k \cos \theta_k) \cos \frac{\Delta\theta_k}{2} = 0. \quad (14)$$

The discrete Euler–Lagrange equations (5) associated with Lagrangian (12) and constraint (8) are

$$\begin{aligned} (J + ma^2)\Delta\theta_k + \frac{ma}{2}\Delta\theta_k \left( \Delta x_k \cos \frac{\theta_{k+1} + \theta_k}{2} + \Delta y_k \sin \frac{\theta_{k+1} + \theta_k}{2} \right) \\ = (J + ma^2)\Delta\theta_{k-1} - \frac{ma}{2}\Delta\theta_{k-1} \left( \Delta x_{k-1} \cos \frac{\theta_k + \theta_{k-1}}{2} + \Delta y_{k-1} \sin \frac{\theta_k + \theta_{k-1}}{2} \right), \end{aligned} \quad (15)$$

$$m\Delta x_k - ma\Delta\theta_k \sin \frac{\theta_{k+1} + \theta_k}{2} = m\Delta x_{k-1} - ma\Delta\theta_{k-1} \sin \frac{\theta_k + \theta_{k-1}}{2} - \lambda_k \sin \theta_k, \quad (16)$$

$$m\Delta y_k + ma\Delta\theta_k \cos \frac{\theta_{k+1} + \theta_k}{2} = m\Delta y_{k-1} + ma\Delta\theta_{k-1} \cos \frac{\theta_k + \theta_{k-1}}{2} + \lambda_k \cos \theta_k. \quad (17)$$

In order to obtain a closed system, these equations should be coupled with the discrete constraint (13).

Eliminating the Lagrange multiplier  $\lambda_k$  from equations (16) and (17), one obtains

$$\begin{aligned} m(\Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k) - ma\Delta\theta_k \sin \frac{\Delta\theta_k}{2} \\ = m(\Delta x_{k-1} \cos \theta_k + \Delta y_{k-1} \sin \theta_k) + ma\Delta\theta_{k-1} \sin \frac{\Delta\theta_{k-1}}{2}. \end{aligned} \quad (18)$$

The dynamics of the discrete Chaplygin sleigh is therefore governed by equations (13), (15), and (18).

## 4.2 The Discrete Momentum Dynamics

As in the continuous-time case, it is instructive to study the dynamics of the nonholonomic momentum. Recall that the nonholonomic momentum for the Chaplygin sleigh has two components—the angular momentum of the sleigh relative to the vertical line through the contact point, and the projection of the linear momentum onto the direction of the blade.

In this paper, the discrete momentum is defined as  $-D_1 L_d(q_k, q_{k+1})$ . Evaluating the derivatives of  $L_d$  and taking into account the constraint (13), the angular component of the discrete momentum is

$$p_1^k = (J + ma^2)\Delta\theta_k + \frac{ma}{2}\Delta\theta_k \left( \Delta x_k \cos \frac{\theta_{k+1} + \theta_k}{2} + \Delta y_k \sin \frac{\theta_{k+1} + \theta_k}{2} \right) \quad (19)$$

and the spatial linear components of the discrete momentum are

$$\mu_x^k = \Delta x_k - ma\Delta\theta_k \sin \frac{\theta_{k+1} + \theta_k}{2}, \quad \mu_y^k = m\Delta y_k + ma\Delta\theta_k \cos \frac{\theta_{k+1} + \theta_k}{2}.$$

One then computes the body component of the linear momentum along the blade,

$$p_2^k = \mu_x^k \cos \theta_k + \mu_y^k \sin \theta_k = m(\Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k) - ma\Delta\theta_k \sin \frac{\Delta\theta_k}{2}. \quad (20)$$

Similar to the continuous-time case, the discrete nonholonomic momentum for the Chaplygin sleigh is defined to be  $p^k = (p_1^k, p_2^k)$ .<sup>3</sup>

**Lemma 1.** *Let the increments  $\Delta x_k$  and  $\Delta y_k$  satisfy the discrete constraint (13). Then*

$$\Delta x_{k-1} \cos \theta_k + \Delta y_{k-1} \sin \theta_k = \Delta x_{k-1} \cos \theta_{k-1} + \Delta y_{k-1} \sin \theta_{k-1}. \quad (21)$$

*Proof.* Using (14), we obtain

$$\begin{aligned} & \Delta x_{k-1} \cos \theta_k + \Delta y_{k-1} \sin \theta_k \\ &= (\Delta x_{k-1} \cos \theta_{k-1} + \Delta y_{k-1} \sin \theta_{k-1}) \cos \Delta\theta_{k-1} \\ & \quad + (-\Delta x_{k-1} \sin \theta_{k-1} + \Delta y_{k-1} \cos \theta_{k-1}) \sin \Delta\theta_{k-1} \\ &= (\Delta x_{k-1} \cos \theta_{k-1} + \Delta y_{k-1} \sin \theta_{k-1}) \left( \cos \Delta\theta_{k-1} + \sin \Delta\theta_{k-1} \tan \frac{\Delta\theta_{k-1}}{2} \right) \\ &= (\Delta x_{k-1} \cos \theta_{k-1} + \Delta y_{k-1} \sin \theta_{k-1}) \left( 1 - 2 \sin^2 \frac{\Delta\theta_{k-1}}{2} + 2 \sin^2 \frac{\Delta\theta_{k-1}}{2} \right) \\ &= \Delta x_{k-1} \cos \theta_{k-1} + \Delta y_{k-1} \sin \theta_{k-1}. \end{aligned}$$

□

**Theorem 4.** *The discrete momentum dynamics is governed by the equations  $p^k = \mathcal{P}(p^{k-1})$ , where the map  $\mathcal{P} : p \mapsto p'$  is defined by*

$$p'_1 = p_1 - a \frac{\Delta\theta}{\cos(\Delta\theta/2)} p_2 - ma^2 \Delta\theta^2 \tan \frac{\Delta\theta}{2}, \quad p'_2 = p_2 + 2ma\Delta\theta \sin \frac{\Delta\theta}{2}, \quad (22)$$

<sup>3</sup>The components of the nonholonomic momentum (9) of the Chaplygin sleigh equal the limits  $\lim_{h \rightarrow 0} p_1^k/h$  and  $\lim_{h \rightarrow 0} p_2^k/h$ , respectively.

with  $\Delta\theta$  satisfying the equation

$$(J + ma^2)\Delta\theta + \frac{a}{2} \frac{\Delta\theta}{\cos(\Delta\theta/2)} \left( p_2 + ma\Delta\theta \sin \frac{\Delta\theta}{2} \right) - p_1 = 0. \quad (23)$$

*Proof.* Using Lemma 1, the discrete dynamics becomes equation (15) coupled with the equation

$$\begin{aligned} m(\Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k) - ma\Delta\theta_k \sin \frac{\Delta\theta_k}{2} \\ = m(\Delta x_{k-1} \cos \theta_{k-1} + \Delta y_{k-1} \sin \theta_{k-1}) + ma\Delta\theta_{k-1} \sin \frac{\Delta\theta_{k-1}}{2}. \end{aligned}$$

One then uses the definitions of  $p_1^k$  and  $p_2^k$  (see formulae (19) and (20)) to obtain equations (22). Equation (23) follows from the formula

$$\begin{aligned} p_1^k = (J + ma^2)\Delta\theta_k + \frac{ma}{2} \left( (\Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k) \cos \frac{\Delta\theta_k}{2} \right. \\ \left. + (-\Delta x_k \sin \theta_k + \Delta y_k \cos \theta_k) \sin \frac{\Delta\theta_k}{2} \right), \end{aligned}$$

the constraint equation (14), and the definition of  $p_2^k$  (formula (20)).  $\square$

### 4.3 The Dynamics of the Balanced Discrete Chaplygin Sleigh

Recall that this case is distinguished by the condition  $a = 0$ . The properties of the discrete dynamics of the balanced sleigh are summarized in the following theorem.

**Theorem 5.** *The discrete dynamics preserves both the discrete nonholonomic momentum and the volume in the momentum plane. The contact point  $(x_k, y_k)$  moves either along a circle or along a straight line at a constant rate.*

*Proof.* If  $a = 0$ , equations (22) imply  $p^k = p^{k-1}$ , which explains the first statement of the theorem. Using (19) and (20) and setting  $a = 0$ , the equations  $p^k = p^{k-1}$  are equivalent to

$$\Delta\theta_k = \text{const}, \quad \Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k = \text{const}.$$

If  $\Delta\theta_k = 0$ , the angle  $\theta_k$  remains the same, and therefore the contact point  $(x_k, y_k)$  moves along a straight line. If  $\Delta\theta_k = \Delta\theta \neq 0$ , the magnitude of the incremental displacement  $(\Delta x_k, \Delta y_k)$  in the  $xy$ -plane equals  $\Delta x_k \cos \theta_k + \Delta y_k \sin \theta_k$ . Therefore, each iteration increases the angular direction of the incremental displacement by  $\Delta\theta$  while preserving the magnitude, which results in the circular motion of the contact point.  $\square$

### 4.4 The Dynamics of the Unbalanced Discrete Chaplygin Sleigh

Recall that the continuous-time dynamics of the unbalanced Chaplygin sleigh is neither momentum nor measure preserving. As equations (22) indicate, the discrete dynamics of the unbalanced Chaplygin sleigh does not conserve the body momentum. Below we assume  $|\Delta\theta_k| < 2\pi$ .

**Theorem 6.** *Map (22) has a one-dimensional manifold of equilibria  $\{p_1 = 0\}$ . The equilibria  $(0, p_2)$  are stable, and asymptotically stable with respect to  $p_1$ , if  $0 < p_2 < 2(J + ma^2)/a$ , and are unstable if  $-2(J + ma^2)/a < p_2 < 0$ .*

*Proof.* Since  $|\Delta\theta| < 2\pi$ , a point  $(p_1, p_2)$  is an equilibrium of (22) if and only if  $\Delta\theta_k = 0$ . Equation (23) then implies that  $p_1 = 0$  while  $p_2$  is arbitrary.

Let  $F(p_1, p_2, \Delta\theta)$  be the left-hand side of (23). The derivative  $\partial F/\partial\Delta\theta$  is computed to be

$$(J + ma^2) + ap_2 \frac{2 \cos(\Delta\theta/2) + \Delta\theta \sin(\Delta\theta/2)}{4 \cos^2(\Delta\theta/2)} + ma^2 \left( \Delta\theta \tan \frac{\Delta\theta}{2} + \frac{\Delta\theta^2}{\cos^2(\Delta\theta/2)} \right).$$

When evaluated at  $\Delta\theta = 0$ , this derivative becomes  $J + ma^2 + ap_2/2$ . The latter quantity is not equal to zero because  $|p_2| < 2(J + ma^2)/a$ . By the implicit function theorem, the equation  $F(p_1, p_2, \Delta\theta) = 0$  defines a function  $\Delta\theta = f(p_1, p_2)$  such that  $f(0, p_2) = 0$ .<sup>4</sup> Using the implicit function theorem, one finds

$$\frac{\partial\Delta\theta}{\partial p_1}(0, p_2) = \frac{2}{2(J + ma^2) + ap_2}, \quad \frac{\partial\Delta\theta}{\partial p_2}(0, p_2) = 0. \quad (24)$$

Next, we compute the linearization of  $\mathcal{P}$  at  $(0, p_2)$ . Evaluating the partial derivatives of formulae (22) and using (24), the linearization of  $\mathcal{P}$  at  $(0, p_2)$  is found to be

$$\begin{pmatrix} 1 - \frac{2ap_2}{2(J + ma^2) + ap_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = 1 - \frac{2ap_2}{2(J + ma^2) + ap_2} \quad \text{and} \quad \lambda_2 = 1.$$

The eigenvalue  $\lambda_1$  is positive and smaller than 1 if  $0 < p_2 < 2(J + ma^2)/a$ , and is greater than 1 if  $-2(J + ma^2)/a < p_2 < 0$ . Observe that the equilibria  $(0, p_2)$  of the map  $\mathcal{P}$  fill out the center manifold  $\{p_1 = 0\}$ . Therefore the equilibria  $(0, p_2)$ ,  $p_2 > 0$ , are located at the intersections of the center manifold and a family of stable manifolds of map (22). Similarly, the equilibria  $(0, p_2)$ ,  $p_2 < 0$ , are located at the intersections of the center manifold and a family of unstable manifolds, which explains the stability statement of the theorem.  $\square$

**Corollary 2.** *The momentum dynamics of the discrete Chaplygin sleigh is phase volume preserving if and only if  $a = 0$ .*

The numerical simulations of the discrete momentum dynamics and the trajectory of the contact point in the unbalanced case are shown in Figures 2 and 3, respectively. The discrete momentum dynamics near the origin of the momentum plane clearly resembles the continuous momentum dynamics (see Figure 1). There certainly is numerical evidence of the existence of discrete heteroclinic trajectories connecting the pairs of unstable equilibria  $(0, -p_2)$  and stable equilibria  $(0, p_2)$ ,  $p_2 > 0$ . The dots in Figure 3 represent the generic discrete trajectory of the contact

<sup>4</sup>The equation  $F(p_1, p_2, \Delta\theta) = 0$  has multiple solutions and therefore defines  $\Delta\theta$  as a multivalued function of  $(p_1, p_2)$ . We choose the branch that passes through the origin in the coordinate space  $(p_1, p_2, \Delta\theta)$  because in the continuous-time case  $\dot{\theta} = 0$  if and only if  $p_1 = 0$ .

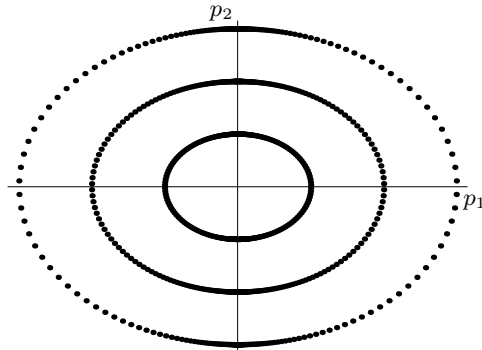


Figure 2: The discrete momentum dynamics near the origin.

point in the  $xy$ -plane (the continuous-time trajectory is the solid curve). The discrete dynamics captures the qualitative behavior of the Chaplygin sleigh's dynamics. In particular, the discrete trajectory has a cusp and asymptotically approaches the straight line motions as  $k \rightarrow \pm\infty$ .

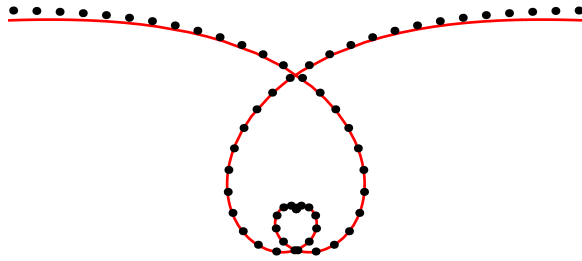


Figure 3: The discrete and continuous trajectories of the contact point.

## 5 Conclusions

This paper introduces and studies the discrete model of the Chaplygin sleigh. The discrete model is shown to mirror the dynamics of the continuous-time model. In particular, the preservation of both the momentum and measure by the discrete flow in the balanced case is observed. The general theory of discrete nonholonomic systems with symmetry and discrete conservation laws will be discussed in future publications.

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## References

- [1] Arnold, V. I., V. V. Kozlov, and A. I. Neishtadt [1989], *Mathematical Aspects of Classical and Celestial Mechanics, Dynamical System III*, Springer-Verlag, New York.
- [2] Bloch, A. M. [2003], *Nonholonomic Mechanics and Control*, Interdisciplinary Applied Mathematics **24**, Springer-Verlag, New York.
- [3] Bloch A. M., P. S. Krishnaprasad, J. E. Marsden, and R. Murray [1996], Nonholonomic Mechanical Systems with Symmetry, *Arch. Rational Mech. Anal.* **136**, 21–99.
- [4] Bobenko A. I. and Y. B. Suris [1999], Discrete Lagrangian Reduction, Discrete Euler–Poincaré Equations, and Semidirect Products, *Lett. Math. Phys.* **49**, 79–93.
- [5] Chaplygin, S. A. [1911], On the Theory of Motion of Nonholonomic Systems. The Theorem on the Reducing Multiplier, *Math. Sbornik XXVIII*, 303–314, (in Russian).
- [6] Cortés J. and S. Martínez [2001], Nonholonomic Integrators, *Nonlinearity* **14**, 1365–1392.
- [7] Fedorov Yu. N. and V. V. Kozlov [1995], Various Aspects of  $n$ -Dimensional Rigid Body Dynamics, *Amer. Math. Soc. Transl.* **168**, 141–171.
- [8] Kozlov, V. V. [1988], Invariant Measures of the Euler–Poincaré Equations on Lie Algebras, *Funct. Anal. Appl.* **22**, 58–59.
- [9] de León, M., D. Martín de Diego, and A. Santamaría Merino [2002], Geometric Integrators and Nonholonomic Mechanics, arXiv:math-ph/0211028.
- [10] Marsden, J. E., S. Pekarsky, and S. Shkoller [1999], Discrete Euler–Poincaré and Lie–Poisson Equations, *Nonlinearity* **12**, 1647–1662.
- [11] Marsden, J. E. and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics **17**, Springer-Verlag, New York.
- [12] Marsden, J. E. and M. West [2001], Discrete Mechanics and Variational Integrators, *Acta Numerica* **10**, 357–514.
- [13] McLachlan, R. [1993], Explicit Lie–Poisson Integration and the Euler Equations, *Phys. Rev. Lett.* **71**, 3043–3046.
- [14] Moser, J. and A. Veselov [1991], Discrete Versions of Some Classical Integrable Systems and Factorization of Matrix Polynomials, *Comm. Math. Phys.* **139**, 217–243.
- [15] Neimark, Ju. I. and N. A. Fufaev [1972] *Dynamics of Nonholonomic Systems*, Translations of Mathematical Monographs **33**, AMS, Providence.
- [16] Veselov, A. P. [1988], Integrable Discrete-Time Systems and Difference Operators, *Funct. Anal. Appl.* **22**, 1–13.

- [17] Veselov, A. P. [1991], Integrable Lagrangian Correspondences and the Factorization of Matrix Polynomials, *Funct. Anal. Appl.* **25**, 38–49.
- [18] Wendland, J. M. and J. E. Marsden [1997], Mechanical Integrators Derived from a Discrete Variational Principle, *Physica D* **106**, 223–246.
- [19] Zenkov, D. V. and A. M. Bloch [2000], Dynamics of the  $n$ -Dimensional Suslov Problem, *J. Geom. Phys.* **34**, 121–136.
- [20] Zenkov, D. V. and A. M. Bloch [2003], Invariant Measures of Nonholonomic Flows with Internal Degrees of Freedom, *Nonlinearity* **16**, 1793–1807.
- [21] Zenkov, D. V. [2003], Linear Conservation Laws of Nonholonomic Systems with Symmetry, *Discrete and Continuous Dynamical Systems (extended volume)*, 963–972.