# Quasivelocities and Stabilization of Relative Equilibria of Underactuated Nonholonomic Systems

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***Abstract*— This paper is concerned with the theory of quasivelocities and its applications to control. The equations of motion of a mechanical system are derived using the Lagrange–d'Alembert principle written in an arbitrary configuration-dependent frame. The structure of the equations of motion written in quasivelocities is utilized in the design of a nonlinear feedback stabilizing controller for an example of a nonholonomically constrained underactuated system.**

## I. INTRODUCTION

Quasivelocities are the velocities of a mechanical system expressed relative to a configuration-dependent frame. The vector fields that form such frames need not be associated with (local) configuration coordinates, so in fact, this paper does not mention "quasicoordinates", a vacuous concept that has led to some confusion in the literature. A good example of quasivelocities is the set of components of the angular velocity of a rigid body rotating about a fixed point.

One of the reasons for using quasivelocities is that the Euler–Lagrange equations written in generalized coordinates are not always effective for analyzing the dynamics of a mechanical system of interest. For example, it is difficult to study the motion of the Euler top if the Euler–Lagrange equations (either intrinsically or in generalized coordinates) are used to represent the dynamics. On the other hand, the use of the angular velocity components relative to a body frame pioneered by Euler [10] results in a much simpler representation of dynamics. Euler's approach was further developed by Lagrange [14] for reasonably general Lagrangians on the rotation group and by Poincaré [21] for arbitrary Lie groups (see [16] for details and history). Other examples include the use of velocity and angular velocity components relative to a moving frame in the study of dynamics of a rigid body moving on a surface as discussed in [22] and [15]. Optimal control of systems using the theory of quasivelocities was discussed in [18].

The goals of this paper are to review the contemporary geometric exposition of quasivelocities and to discuss the use of quasivelocities in the design of stabilizing controllers. The equations of motion written in terms of quasivelocities are called the ***Hamel equations***. Their derivation originated in Poincaré [21] and Hamel [11]. The Hamel equations are derived in this paper from a variational point of view; that is, a form of the principle of critical action is developed that is equivalent to the Hamel equations. The motivation for studying this variational approach is that we expect the variational structure to be useful in the development of discrete-time models that approximate the continuous-time mechanics (see e.g. [17]). The Hamel equations are also particularly useful in the analysis of nonholonomically constrained systems (see [1] and [7]).

The directions of the control inputs in underactuated systems form a subbundle of the system's momentum phase space that may fail to be a cotangent bundle to a submanifold of the configuration space. This is typical if control torques are used. In such situations a suitable choice of a frame may result in a simpler representation of the controlled dynamics. Moreover, it may assist in the selection of stabilizing control inputs. We illustrate this approach in this paper by analyzing the problem of stabilization of the slow upright uniform motions of a disk. Specifically, we show that an energy-momentum based method for the analysis of the stability of nonholonomic systems as developed in [25] may be used to analyze stabilization in the underactuated setting. For related earlier work see [3], [8], and references therein.

## II. LAGRANGIAN MECHANICS

### *A. The Euler–Lagrange Equations*

A Lagrangian mechanical system is specified by a smooth manifold $Q$ called the ***configuration space*** and a function $L : TQ \to \mathbb{R}$ called the ***Lagrangian***. In many cases, the Lagrangian is the kinetic minus potential energy of the system with the kinetic energy defined by a Riemannian metric on the configuration manifold and the potential energy being a smooth function on $Q$. If necessary, non-conservative forces can be introduced (e.g., gyroscopic forces that are represented by terms in $L$ that are linear in the velocity), but this is not discussed in detail in this paper.

In local coordinates $q = (q^1, \dots, q^n)$ on the configuration space $Q$ we write $L = L(q, \dot q)$. The dynamics is given by the ***Euler–Lagrange equations***

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, \dots, n. \tag{1}$$

These equations were originally derived by Lagrange [14] in 1788 by requiring that simple force balance $\boldsymbol{F} = m\boldsymbol{a}$ be *covariant*, *i.e.* expressible in arbitrary generalized coordinates. A variational derivation of the Euler–Lagrange equations, namely Hamilton's principle (also called the principle of critical action), came later in the work of Hamilton [12]

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and [13] in 1834/35. For more details see [1], [16], and theorem 2.1 below.

### B. The Hamel Equations

In this paragraph we introduce the Hamel equations. In §II-C we derive these equations from a global variational principle, generalizing the reduced principle of critical action for systems with symmetry.

In many cases the Lagrangian and the equations of motion have a simpler structure when written using so-called noncommuting variables. An example of such a system is the rigid body. Below we develop a general approach that allows one to obtain the equations of motion in such noncommuting variables.

Let $q = (q^1, \ldots, q^n)$ be local coordinates on the configuration space $Q$ and $u_i \in TQ$, $i = 1, \ldots, n$, be smooth independent *local* vector fields defined in the same coordinate neighborhood.[1] The components of $u_i$ relative to the basis $\partial/\partial q^j$ will be denoted $\psi_i^j$; that is,

$$u_i(q) = \psi_i^j(q) \frac{\partial}{\partial q^j}, \quad i, j = 1, \ldots, n,$$

where a sum on $j$ is understood.

Let $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$ be the components of the velocity vector $\dot{q} \in TQ$ relative to the basis $u_1, \ldots, u_n$, *i.e.*,

$$\dot{q} = v^i u_i(q); \tag{2}$$

then

$$l(q, v) := L(q, v^i u_i(q))$$

is the Lagrangian of the system written in the local coordinates $(q, v)$ on the tangent bundle $TQ$. The coordinates $(q, v)$ are Lagrangian analogues of non-canonical variables in Hamiltonian dynamics.

Define the quantities $c_{il}^m(q)$ by the equations

$$[u_i(q), u_l(q)] = c_{il}^m(q) u_m(q), \quad i, l, m = 1, \ldots, n. \tag{3}$$

These quantities vanish if and only if the vector fields $u_i(q)$, $i = 1, \ldots, n$, commute. One finds that

$$c_{il}^m = (\psi^{-1})_k^m \left( \frac{\partial \psi_l^k}{\partial q^j} \psi_i^j - \frac{\partial \psi_i^k}{\partial q^j} \psi_l^j \right).$$

Given two elements $v, w \in \mathbb{R}^n$, define the antisymmetric bracket operation $[\cdot\,,\cdot]_q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$[v, w]_q = [v^i u_i, w^j u_j](q)$$

where $[\cdot\,,\cdot]$ is the Jacobi–Lie bracket of vector fields on $Q$.

Therefore, each tangent space $T_qQ$ is isomorphic to the Lie algebra $V_q := (\mathbb{R}^n, [\cdot\,,\cdot]_q)$. Thus, if the fields $u_1, \ldots, u_n$ are independent in $U \subset Q$, the tangent bundle $TU$ is diffeomorphic to a Lie algebra bundle over $U$.

The ***dual*** of $[\cdot\,,\cdot]_q$ is, by definition, the operation $[\cdot\,,\cdot]_q^* : V_q \times V_q^* \to V_q^*$ given by

$$\langle [v, \alpha]_q^*, w \rangle := \langle \alpha, [v, w]_q \rangle.$$

[1]In certain cases, some or all of $u_i$ can be chosen to be *global* vector fields on $Q$.

Using (3), the coordinate representations of $[\cdot\,,\cdot]_q$ and $[\cdot\,,\cdot]_q^*$ are computed to be

$$\left([v, w]_q\right)^m = c_{ij}^m v^i w^j \quad \text{and} \quad \left([v, \alpha]_q^*\right)_j = c_{ij}^m v^i \alpha_m.$$

Let $u = (u_1, \ldots, u_n) \in TQ \times \cdots \times TQ$. For a function $f : Q \to \mathbb{R}$, define $u[f] \in V_q^*$ by $u[f] = (u_1[f], \ldots, u_n[f])$, where $u_i[f] = \psi_i^j \partial_j f$ is the usual directional derivative of $f$ along the vector field $u_i$. Viewing $u_i$ as vector fields on $TQ$ whose fiber components equal 0 (that is, taking the vertical lift of these vector fields), one defines the directional derivatives $u_i[l]$ for a function $l : TQ \to \mathbb{R}$ by the formula

$$u_i[l] = \psi_i^j \frac{\partial l}{\partial q^j}.$$

The evolution of the variables $(q, v)$ is governed by the ***Hamel equations***

$$\frac{d}{dt} \frac{\partial l}{\partial v} = \left[ v, \frac{\partial l}{\partial v} \right]_q^* + u[l] \tag{4}$$

coupled with (2). In (4), $u[l] = (u_1[l], \ldots, u_n[l])$. If $u_i = \partial/\partial q^i$, equations (4) become the Euler–Lagrange equations (1). The coordinate form of equations (4) is

$$\frac{d}{dt} \frac{\partial l}{\partial v^i} = c_{ji}^m \frac{\partial l}{\partial v^m} v^j + u_i[l].$$

Equations (4) were introduced in [11] (see also [19] for details and some history).

### C. The Principle of Critical Action

Let $\gamma : [a, b] \to Q$ be a smooth curve in the configuration space. A ***variation*** of the curve $\gamma(t)$ is a smooth map $\beta : [a, b] \times [-\varepsilon, \varepsilon] \to Q$ that satisfies the condition $\beta(t, 0) = \gamma(t)$. This variation defines the vector field

$$\delta\gamma(t) = \left. \frac{\partial \beta(t, s)}{\partial s} \right|_{s=0}$$

along the curve $\gamma(t)$.

*Theorem 2.1: Let $L : TQ \to \mathbb{R}$ be a Lagrangian and $l : TQ \to \mathbb{R}$ be its representation in local coordinates $(q, v)$. Then, the following statements are equivalent:*

(i) *The curve $q(t)$, where $a \le t \le b$, is a critical point of the action functional*

$$\int_a^b L(q, \dot{q})\, dt \tag{5}$$

*on the space of curves $\Omega(Q; q_a, q_b)$ in $Q$ connecting $q_a$ to $q_b$ on the interval $[a, b]$, where we choose variations of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$.*

(ii) *The curve $q(t)$ satisfies the Euler–Lagrange equations (1).*

(iii) *The curve $(q(t), v(t))$ is a critical point of the functional*

$$\int_a^b l(q, v)\, dt \tag{6}$$

*with respect to variations $\delta v$, induced by the variations $\delta q = w^i u_i(q)$, and given by*

$$\delta v = \dot{w} + [v, w]_q.$$

(iv) *The curve* $(q(t), v(t))$ *satisfies the Hamel equations* (4) *coupled with the equations* $\dot{q} = \langle u(q), v\rangle \equiv v^i u_i(q)$.

For the early development of these equations see [21] and [11].

*Proof:* The equivalence of (i) and (ii) is standard and is proved by computing the variational derivative of the action functional (5),

$$\delta \int_a^b L(q, \dot{q})\, dt = \int_a^b \left( \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt$$
$$= \int_a^b \left( \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt.$$

Denote the components of $\delta q(t)$ relative to the basis $u_1(q(t)), \ldots, u_n(q(t))$ by $w(t) = (w^1(t), \ldots, w^n(t))$, that is,

$$\delta q(t) = \langle u, w\rangle \equiv w^i(t) u_i(q(t)).$$

To prove the equivalence of (i) and (iii), we first compute the quantities $\delta \dot{q}$ and $d(\delta q)/dt$:

$$\delta \dot{q} = \delta \left[ v^i(t) u_i(q(t)) \right] = \delta v^i(t) u_i(q(t)) + v^i(t) \frac{\partial u_i}{\partial q^j} \delta q^j,$$
$$\frac{d(\delta q)}{dt} = \frac{d}{dt}\left( w^i(t) u_i(q(t)) \right) = \dot{w}^i(t) u_i(q(t)) + w^i(t) \frac{\partial u_i}{\partial q^j} \dot{q}^j.$$

Since $\delta \dot{q} = d(\delta q)/dt$, we obtain

$$\delta v^i(t) u_i(q(t)) = \left( \dot{w}^i(t) + c^i_{kl}(q(t)) v^k(t) w^l(t) \right) u_i(q(t));$$

that is,[2]

$$\delta v(t) = \dot{w}(t) + [v(t), w(t)]_{q(t)}.$$

To prove the equivalence of (iii) and (iv), we use the above formula and compute the variational derivative of the functional (6):

$$\delta \int_a^b l(q, v)\, dt = \int_a^b \left( \frac{\partial l}{\partial q}\delta q + \frac{\partial l}{\partial v}\delta v \right) dt$$
$$= \int_a^b \left( \frac{\partial l}{\partial q} w^i u_i + \frac{\partial l}{\partial v}\left( \dot{w} + [v, w]_{q(t)} \right) \right) dt$$
$$= \int_a^b \left( u[l] + \left[ v, \frac{\partial l}{\partial v} \right]^*_{q(t)} - \frac{d}{dt}\frac{\partial l}{\partial v} \right) w\, dt.$$

This variational derivative vanishes if and only if the Hamel equations are satisfied. ∎

## III. Nonholonomic Systems

### A. The Lagrange–d'Alembert Principle

Assume now that there are *velocity constraints* imposed on the system. We confine our attention to constraints that are homogeneous in the velocity. Accordingly, we consider a configuration space $Q$ and a distribution $\mathcal{D}$ on $Q$ that describes these constraints. Recall that a distribution $\mathcal{D}$ is a collection of linear subspaces of the tangent spaces of $Q$; we denote these spaces by $\mathcal{D}_q \subset T_q Q$, one for each $q \in Q$. A curve $q(t) \in Q$ will be said to ***satisfy the constraints*** if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all $t$. This distribution will, in general, be nonintegrable; *i.e.,* the constraints are, in general, nonholonomic.[3]

Consider a Lagrangian $L : TQ \to \mathbb{R}$. In coordinates $q^i, i = 1, \ldots, n$, on $Q$ with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we write $L(q^i, \dot{q}^i)$. The equations of motion are given by the following Lagrange–d'Alembert principle.

*Definition 3.1: The* ***Lagrange–d'Alembert equations of motion*** *for the system are those determined by*

$$\delta \int_a^b L(q, \dot{q})\, dt = 0,$$

*where we choose variations* $\delta q(t)$ *of the curve* $q(t)$ *that satisfy* $\delta q(a) = \delta q(b) = 0$ *and* $\delta q(t) \in \mathcal{D}_{q(t)}$ *for each* $t$ *where* $a \le t \le b$.

This principle is supplemented by the condition that the curve $q(t)$ itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations (see [4] for a discussion and references).

One way to write the dynamics is to make use of the *Euler–Lagrange equations with multipliers.* This is done below in coordinates.

The distribution $\mathcal{D}$ can be locally written as

$$\mathcal{D} = \{ \dot{q} \in TQ \mid A^s_i(q)\dot{q}^i = 0,\ s = 1, \ldots, p \}.$$

The ***constrained variations*** $\delta q(t) \in TQ$ satisfy the equations

$$A^s_i(q)\delta q^i = 0, \quad s = 1, \ldots, p. \tag{7}$$

Using the Lagrange–d'Alembert principle and (7), one writes the equations of motion with *Lagrange multipliers* as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} + \lambda_s A^s_i(q), \quad A^s_i(q)\dot{q}^i = 0.$$

The presence of the Lagrange multipliers in the equations of motion is not always desirable. One way to eliminate the Lagrange multipliers in the nonholonomic setting is to use the Hamel equations, as shown below.

### B. The Constrained Hamel Equations

Given a nonholonomic system, that is, a Lagrangian $L : TQ \to \mathbb{R}$ and constraint distribution $\mathcal{D}$, select the independent (local) vector fields

$$u_i : Q \to TQ, \quad i = 1, \ldots, n,$$

such that $\mathcal{D}_q = \operatorname{span}\{u_1(q), \ldots, u_{n-p}(q)\}$. Each $\dot{q} \in TQ$ can be uniquely written as

$$\dot{q} = \langle u(q), v^{\mathcal{D}}\rangle + \langle u(q), v^{\mathcal{U}}\rangle, \ \text{ where } \ \langle u(q), v^{\mathcal{D}}\rangle \in \mathcal{D}_q, \tag{8}$$

*i.e.,* $\langle u(q), v^{\mathcal{D}}\rangle$ is the component of $\dot{q}$ along $\mathcal{D}_q$. Similarly, each $a \in T^*Q$ can be uniquely decomposed as

$$a = \langle a_{\mathcal{D}}, u^*(q)\rangle + \langle a_{\mathcal{U}}, u^*(q)\rangle,$$

[2] If $Q$ is a Lie group, this formula is derived in [5].

[3] Constraints are nonholonomic if and only if they cannot be rewritten as *position* constraints.

where $\langle a_{\mathcal{D}}, u^*(q)\rangle$ is the component of $a$ along the dual of $\mathcal{D}_q$, and where $u^*(q) \in T^*Q \times \cdots \times T^*Q$ denotes the dual frame of $u(q)$. Using (8), the constraints read

$$v = v^{\mathcal{D}} \quad \text{or} \quad v^{\mathcal{U}} = 0. \tag{9}$$

This implies

$$\delta v = \delta v^{\mathcal{D}} \quad \text{or} \quad \delta v^{\mathcal{U}} = 0. \tag{10}$$

Using the Lagrange–d'Alembert principle and (10) proves the following theorem:

*Theorem 3.2: The dynamics of a nonholonomic system is represented by the* ***constrained Hamel equations***

$$\left(\frac{d}{dt}\frac{\partial l}{\partial v} - \left[v^{\mathcal{D}}, \frac{\partial l}{\partial v}\right]_q^* - u[l]\right)_{\mathcal{D}} = 0 \tag{11}$$

*coupled with the constraint equation* (9) *and the kinematic equation*

$$\dot{q} = \langle u(q), v^{\mathcal{D}}\rangle. \tag{12}$$

In coordinate notation, equations (11) read as follows:

$$\frac{d}{dt}\frac{\partial l}{\partial v^i} = c_{ji}^m \frac{\partial l}{\partial v^m} v^j + u_i[l], \quad i, j = 1, \ldots, n-p.$$

### C. Chaplygin Sleigh

In order to illustrate the difference between holonomic and nonholonomic dynamics, we describe here the *Chaplygin sleigh*. The Chaplygin sleigh is introduced in [9] and discussed for example in [19]. See also the paper [23], where an interesting connection with systems with impacts is made.

The sleigh is essentially a flat rigid body in the plane supported at three points, two of which slide freely without friction while the third is a knife edge constraint which allows no motion perpendicular to its edge.

We will derive the dynamics using equations (11). Let $\theta$ be the angular orientation of the sleigh and $(x, y)$ be the coordinates of the contact point as shown in Figure 1.

Define the vector fields $u_i$ by the formulae $u_1 = \partial_\theta$, $u_2 = \cos\theta\,\partial_x + \sin\theta\,\partial_y$, $u_3 = -\sin\theta\,\partial_x + \cos\theta\,\partial_y$ (see Figure 1). The velocity components relative to the frame $u_1$, $u_2$, $u_3$ are $v^1 = \dot{\theta}$, $v^2 = \dot{x}\cos\theta + \dot{y}\sin\theta$, and $v^3 = -\dot{x}\sin\theta + \dot{y}\cos\theta$. Thus $v^1$ is is the angular velocity of the sleigh relative to the vertical line through the contact point, and $v^2$ and $v^3$ are the components of linear velocity of the contact point in the directions along and orthogonal to the blade, respectively.

Let the center of mass $C$ be located on the line through the blade at the point $au_2$ relative to the body frame. See Figure 1 for details. Denote the mass and the moment of inertia of the sleigh by $m$ and $J$. The constraint reads $v^3 = 0$.

The Lagrangian is just the kinetic energy of the sleigh,

$$l(v) = \tfrac{1}{2}\left((J + ma^2)(v^1)^2 + m((v^2)^2 + (v^3)^2 + 2av^1v^3)\right),$$

which is the sum of the kinetic energies of the linear and rotational modes of the body.

Equations (11), written for the Chaplygin sleigh, become

$$(J + ma^2)\dot{v}^1 = -ma\,v^1v^2, \quad m\dot{v}^2 = ma(v^1)^2.$$

These equations decouple from the full system (11) and (12).

As shown in [19], a generic trajectory of the contact point of the blade and the plane is either a doubly-asymptotic curve with a cusp that approaches straight lines as $t \to \pm\infty$ like in Figure 2 if $a \neq 0$, or a circle if $a = 0$. The condition $a = 0$ means that the center of mass is situated at the contact point of the sleigh and the plane. Nongeneric trajectories of the contact point are motions along a straight line at a constant speed.

Thus the center of mass of the Chaplygin sleigh almost never moves along a straight line. This dynamics is entirely different from the dynamics of an unconstrained flat rigid body sliding without friction on a horizontal plane where the center of mass is known to move along a straight line.

The structure of generic trajectories of the Chaplygin sleigh enables one to steer the sleigh by means of a moving mass as shown in [20].

## IV. STABILIZATION OF RELATIVE EQUILIBRIA OF NONHOLONOMIC SYSTEMS

As discussed in [1], [4], and [7], nonholonomic constraints and symmetry each define subbundles of the velocity phase space $TQ$ of the system. The base space of both subbundles is the configuration manifold $Q$. We denote the constraint subbundle by $\mathcal{D}$. The intersection of the symmetry subbundle with $\mathcal{D}$ is denoted $\mathcal{S}$. We assume that the dimension of the fibers of $\mathcal{S}$ is positive. The frame $u_i$, $i = 1, \ldots, n$, is usually selected in such a way that there are two subframes that span the fibers of $\mathcal{D}$ and $\mathcal{S}$.

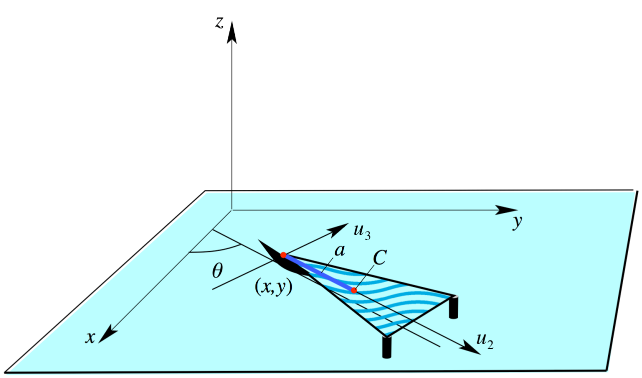

Fig. 1. The Chaplygin sleigh.

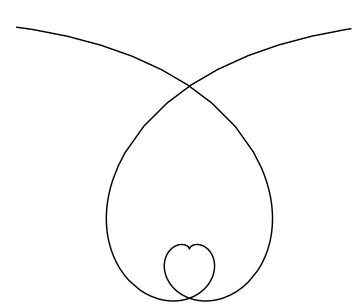
Fig. 2. A typical trajectory of the contact point of the Chaplygin sleigh.

For underactuated systems, the controlled directions are characterized by the fibers of a subbundle $\mathcal{T}^*$ of the momentum phase space $T^*Q$. We assume in this paper that $\mathcal{T}^* \subset \mathcal{S}^*$, where $\mathcal{S}^* \subset T^*Q$ is the bundle over $Q$ whose fibers are the duals of the fibers of $\mathcal{S}$. That is, the control acts in some or all of the symmetry directions consistent with constraints. In this setting, one may select a frame that contains a subframe whose dual spans the fibers of $\mathcal{T}^*$.

In the rest of the paper we assume that the Lagrangian equals the kinetic minus potential energy of the system, and that the kinetic energy is given by a Riemannian metric on the configuration manifold $Q$.

We consider the problem of stabilization of relative equilibria of underactuated nonholonomic systems with symmetry that satisfy the conditions listed above. Under these assumptions, the right-hand sides of the uncontrolled equations of motion on the constrained symmetry subbundle $\mathcal{S}$ are homogeneous quadratic polynomials in quasivelocities. Our strategy is to assign controls that have the same structure, *i.e.,* are given by quadratic polynomials in quasivelocities, and to require that the controlled equations on $\mathcal{S}$ are equivalent to conservation laws. We then utilize the remaining freedom in the control selection and make the equilibria of the system's dynamics, reduced to the levels of the controlled conservation laws, stable. The suggested procedure is motivated by the fact that this reduction is related to the energy-momentum method for stability analysis (see [25]).

The general theory will be discussed in a forthcoming publication. In the next section we use our approach and stabilize slow vertical motions of a rolling disk.

## V. STABILIZATION OF A FALLING DISK

Consider a uniform disk rolling without sliding on a horizontal plane. It is well-known that some of the steady state motions are the uniform motions of a disk along a straight line. Such motions are unstable if the angular velocity of the disk is small. Stability is observed if the angular velocity of the disk exceeds a certain critical value, see [19] and [7] for details. Below we use a steering torque for stabilization of slow unstable motions of the disk.

We assume that the disk has a unit mass and a unit radius. The moments of inertia of the disk relative to its diameter and to the line orthogonal to the disk and through its center are $A$ and $B$, respectively. The configuration coordinates for the disk are $(\theta, \phi, \psi, x, y)$ as in Figure 3. Following [19], we select $u_1$ to be the vector in the $xy$-plane and tangent to the rim of the disk, $u_2$ to be the vector from the contact point to the center of the disk, and $u_3$ to be $u_1 \times u_2$, as shown in Figure 3. In agreement with our general frame selection process, the fields $u_1$, $u_2$, and $u_3$ span the fibers of the constraint distribution, the fields $u_2$ and $u_3$ span the constrained symmetry directions, and the dual of $u_2$ spans the control subbundle. The component of disk's angular velocity along $u_1$ equals $\dot{\theta}$, the $u_2$ and $u_3$ components are denoted by $\xi$ and $\eta$.

Using this frame, the Hamel equations of motion are

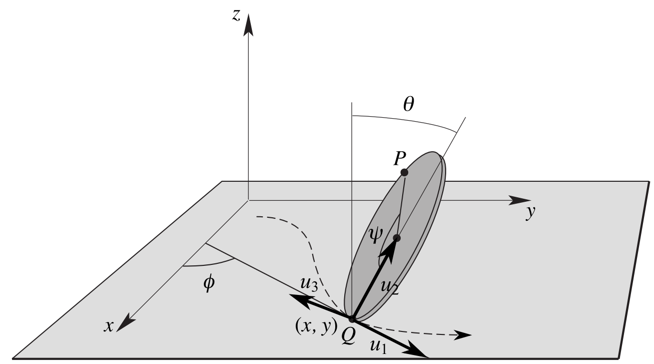


Fig. 3. The geometry of the rolling disk.

computed to be

$$(A+1)\ddot{\theta} + A\xi^2 \tan\theta - (B+1)\xi\eta - g\sin\theta = 0, \quad (13)$$

$$A\dot{\xi} - A\xi\dot{\theta}\tan\theta + B\eta\dot{\theta} = u, \quad (14)$$

$$(B+1)\dot{\eta} + \xi\dot{\theta} = 0, \quad (15)$$

where $u$ is the steering torque and $g$ is the acceleration of gravity. In the absence of the torque, the last two equations can be written as conservation laws of the form

$$\xi = F(\theta, a, b), \quad \eta = G(\theta, a, b); \quad (16)$$

here and below the parameters $a$ and $b$ label the levels of these conservation laws. These conservation laws are obtained by integrating the equations

$$A\frac{d\xi}{d\theta} = A\xi\tan\theta - B\eta, \quad (B+1)\frac{d\eta}{d\theta} = -\xi.$$

Formulae (16) may be interpreted as momentum conservation laws (see [7] and [24]).

Now consider a steady state motion $\theta = 0$, $\xi = 0$, $\eta = \eta_e$. This motion is unstable if $\eta_e$ is small. Set

$$u = -f(\theta)\eta\dot{\theta}, \quad (17)$$

where $f(\theta)$ is a differentiable function. The motivation for the choice (17) for $u$ is that it preserves the structure of equations (14) and (15), and thus the controlled system will have conservation laws whose structure is similar to that of the uncontrolled system. Viewing $\theta$ as an independent variable, we replace equations (14) and (15) with the linear system

$$A\frac{d\xi}{d\theta} = A\xi\tan\theta - (B + f(\theta))\eta, \quad (B+1)\frac{d\eta}{d\theta} = -\xi. \quad (18)$$

The general solution of (18),

$$\xi = F_c(\theta, a, b), \quad \eta = G_c(\theta, a, b), \quad (19)$$

is interpreted as the *controlled conservation laws*. The functions that define these conservation laws are typically difficult or impossible to find explicitly.

Next, we reduce the dynamics to the common level set of the controlled conservation laws (19). From (13), this defines a family of one degree of freedom Lagrangian (or Hamiltonian) systems

$$(A+1)\ddot{\theta} + AF_c^2(\theta, a, b)\tan\theta \\ - (B+1)F_c(\theta, a, b)G_c(\theta, a, b) - g\sin\theta = 0.$$

The stability of the relative equilibrium $\theta = 0$, $\xi = 0$, $\eta = \eta_e$ is tested using the nonholonomic energy-momentum method of [25]. This method requires that

$$\frac{d}{d\theta}\big(AF_c^2(\theta, a_e, b_e)\tan\theta \\ - (B+1)F_c(\theta, a_e, b_e)G_c(\theta, a_e, b_e) - g\sin\theta\big)\Big|_{\theta=0} > 0,$$

where $a_e$ and $b_e$ are defined by the equations

$$F_c(0, a_e, b_e) = 0, \quad G_c(0, a_e, b_e) = \eta_e.$$

This stability condition is obtained by constructing a suitable Lyapunov function, see [25] for details.

Using (18), the stability condition becomes

$$f(0) > \frac{Ag}{(B+1)\eta_e^2} - B. \tag{20}$$

That is, any function $f(\theta)$ whose value at $\theta = 0$ satisfies inequality (20) defines a stabilizing steering torque.

Observe that in the settings considered in the present paper the energy-momentum method gives conditions for nonlinear Lyapunov (nonasymptotic) stability. Hence stabilization by the torque (17) is nonlinear and nonasymptotic. Asymptotic stabilization can be achieved by adding dissipation-emulating terms to the control input.

## VI. CONCLUSIONS AND FUTURE WORK

We have discussed the use of quasivelocities and the associated Hamel equations in the analysis and design of feedback stabilizing controllers for nonholonomic systems. The procedure outlined here enabled us to use an energy-momentum based technique for proving stabilization. In a forthcoming publication we intend to further develop the proposed techniques and to relate them to the method of controlled Lagrangians [2], [6], and [26].

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## REFERENCES

[1] A. M. Bloch, *Nonholonomic Mechanics and Control*, ser. Interdisciplinary Appl. Math. Springer-Verlag, New York, 2003, vol. 24.

[2] A. M. Bloch, D-E. Chang, N. Leonard, and J. E. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems II: Potential shaping," *Trans. IEEE on Autom. Contr.*, vol. 46, pp. 1556–1571, 2001.

[3] A. M. Bloch and P. E. Crouch, "Nonholonomic control systems on Riemannian manifolds," *SIAM J. on Control and Optimization*, vol. 33, pp. 126–148, 1995.

[4] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. Murray, "Nonholonomic mechanical systems with symmetry," *Arch. Rational Mech. Anal.*, vol. 136, pp. 21–99, 1996.

[5] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu, "The Euler–Poincaré equations and double bracket dissipation," *Comm. Math. Phys.*, vol. 175, pp. 1–43, 1996.

[6] A. M. Bloch, N. Leonard, and J. E. Marsden, "Controlled Lagrangians and the stabilization of mechanical systems I: The first matching theorem," *IEEE Trans. on Aut. Control*, vol. 45, pp. 2253–2270, 2000.

[7] A. M. Bloch, J. E. Marsden, and D. V. Zenkov, "Quasivelocities and symmetries in nonholonomic systems," *Dynamical Systems: An International Journal*, vol. 24, pp. 187–222, 2009.

[8] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch, "Control and stabilization of nonholonomic systems," *IEEE Trans. on Aut. Control*, vol. 37, 1746–1757, 1992.

[9] S. A. Chaplygin, "On the theory of motion of nonholonomic systems. Theorem on the reducing multiplier," *Mat. Sbornik*, vol. 28, pp. 303–314, 1911.

[10] L. Euler, "Decouverte d'un nouveau principe de Mecanique," *Mémoires de l'académie des sciences de Berlin*, vol. 6, pp. 185–217, 1752.

[11] G. Hamel, "Die Lagrange–Eulersche Gleichungen der Mechanik," *Z. Math. Phys.*, vol. 50, pp. 1–57, 1904.

[12] W. R. Hamilton, "On a general method in dynamics, Part I," *Phil. Trans. Roy. Soc. Lond.*, pp. 247–308, 1834.

[13] ——, "On a general method in dynamics, Part II," *Phil. Trans. Roy. Soc. Lond.*, pp. 95–144, 1835.

[14] J. L. Lagrange, *Mécanique Analytique*. Chez la Veuve Desaint, 1788.

[15] A. P. Markeev, *Dynamics of a Body Being Contiguous to a Rigid Surface*. Moscow: Nauka, 1992.

[16] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry*, ser. Texts in Appl. Math. New York: Springer-Verlag, 1999, vol. 17.

[17] J. E. Marsden and M. West, "Discrete mechanics and variational integrators," *Acta Numerica*, vol. 10, pp. 357–514, 2001.

[18] J. Maruskin and A. Bloch, "The Boltzman–Hamel equations for optimal control," *Proc. CDC*, vol. 46, pp. 554–559, 2007.

[19] Ju. I. Neimark and N. A. Fufaev, *Dynamics of Nonholonomic Systems*, ser. Translations of Mathematical Monographs. Providence, Rhode Island: AMS, 1972, vol. 33.

[20] J. M. Osborne and D. V. Zenkov, "Steering the Chaplygin sleigh by a moving mass," *Proc. CDC*, vol. 44, pp. 1114–1118, 2005.

[21] H. Poincaré, "Sur une forme nouvelle des équations de la mécanique," *CR Acad. Sci.*, vol. 132, pp. 369–371, 1901.

[22] E. J. Routh, *Treatise on the Dynamics of a System of Rigid Bodies*. MacMillan, London, 1860.

[23] A. Ruina, "Non-holonomic stability aspects of piecewise nonholonomic systems," *Reports in Mathematical Physics*, vol. 42, pp. 91–100, 1998.

[24] D. V. Zenkov, "Linear conservation laws of nonholonomic systems with symmetry," *Disc. Cont. Dyn. Syst.*, extended volume, pp. 963–972, 2003.

[25] D. V. Zenkov, A. M. Bloch, and J. E. Marsden, "The energy-momentum method for the stability of nonholonomic systems," *Dynamics and Stability of Systems*, vol. 13, pp. 128–166, 1998.

[26] ——, "Flat nonholonomic matching," *Proc. ACC*, 2002, 2812–2817.