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Measuring non-stationary cycles: a time-deformation approach.

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Abstract

Many economic variables evolve in a pro-cyclical manner with cycles varying in length and being independent of the calendar time. Such time series appear to be covariance non-stationary in the regular time scale. Therefore, carrying out estimations on the data taken at the regular intervals of time (days, months, years) may undermine important relations between variables or provide improper results.

This paper considers the non-linear deformation of time scale for the G(λ)-stationary processes developed by Jiang, Gray and Woodward (2006). After the appropriate Box-Cox transformation, processes, which are non-stationary in the regular time domain, become stationary in the transformed time scale, thus allowing application of traditional econometric tools.

As an empirical illustration, the cyclical behavior of the U.S. GDP and unemployment series is studied in the context of a structural time series models with explicit trend and cycle modeling.

Introduction

The evolution path of many pro-cyclical economic time series may change irregularly since they depend on underlying fundamentals striking not systematically. Theoretical explanations of this fact account for several models such as modeling the behavior of risk-averse agents (Chalkley and Lee, 2002), intertemporal increasing returns (Acemoglua and Scott, 1997), and adjustment costs model (Caballero and Engel, 1991). At the same time, empirical research in the given area is more extensive. The change in the volatility and cycle duration of major macro variables was reported by many recent studies. Stock and Watson (2003), Kim, Nelson and Piger (2004) found that cyclical component of real GDP became less volatile in the post-war period. Moreover, decrease in volatility is present also in many production sectors of real GDP as well as in final sales. At the same time, inflation is also marked by changes in persistence and volatility at the corresponding periods of the volatility reduction observed in real GDP. The break in conditional volatility and conditional mean of 214 monthly US variables over the 40 years is documented by Sensier and van Dijk (2004). These findings also suggest that periodic economic data such as GDP, unemployment, inflation series, as well as many others, are characterized by different timing of upturns and downturns. At the same time, most of empirical research is carried out on the data taken at regular intervals of time (days, months, years), which may therefore undermine important relations between variables or provide improper results. Therefore, it it is appealing to treat data on a time-scale specific to the process, rather then on the regular time scale. The former may differ significantly from usual time measurement. This paper considers the methodology of transforming the time scale into the one, on which data would become covariance stationary and suitable for econometric estimation.

The discussion of non-stationarity of economic variables begins with the seminal paper by Burns and Mitchell (1946) who suggested that a unit of economic time is defined by business cycle length rather then calendar time. This work initiated a numerous empirical research intended to understand economic fluctuations, which can be divided into a few distinct groups. One group of works deals with modification of existing econometric models (e.g. ARIMA) to account for different behavior of the data over the cycle. An example would be studies of asymmetrical behavior of cycles over the periods of expansion and recession. Neftci (1984) considered estimation and testing issues in the model when underlying data are characterized by downturns and expansions of different time length. Introducing an indicator variable for ups and downs into a linear model can potentially improve model fit and forecasting. The major weakness of works on asymmetry is considering the whole data series as a stationary process. While downturns and expansions are treated differently, the length of cycles is assumed to remain constant over the time.

Another kind of improvement in this field is provided by Markov-switching common factor models, which allow modeling regime changes in the dynamics of cycles. Change in the growth rate of output is treated as discrete regime change from high to low state (Hamilton (1989, 1990), Diebold and Rudebusch (1996)). Kim and Nelson (1998) introduced dynamic factor model with regime switching into the state space framework. Authors modeled the probability of regime change based on the length of recession or expansion phase of business cycle. In addition to regime switching in output growth, Kim and Nelson (1999) also explored the issue of shifts in the parameters of models based on Markov-switching behavior.

A completely different way of research concentrates on developing new econometric tools, such as windowed Fourier transformation, band-pass filtering in the frequency domain (Hodrick and Prescott (1997), Baxter and King (1999)), and wavelet analysis (Yogo (2003), Raihan, Wen, and Zeng (2005)) for analyzing data of varying frequency. All these methods concentrate on frequency domain, rather then on time domain, to explore dynamic properties of the data. However, these tools suffer from some limitations. Fourier transformation is not applicable for nonstationary signals. Wavelet methods, while nicely capturing data behavior, often lack economic interpretation. Filtering methods usually can not be used for forecasting methods, and as shown by Harvey (1993), may provide spurious cyclical behavior.

Finally, one more approach is to modify existing data to obtain required properties for econometric models. The non-linear transformation of the time scale can change the data from non-stationary in regular time to stationary process in modified scale. Stock (1987) introduced the idea of "economic time", which may be different from regular time. He proposed several non-linear transformations connecting calendar time and economic time for cyclical data. One was the extension of work by Burns and Mitchell (1946) based on the phase-averaging procedure. The expansion and contraction of a cycle were split into several phases and then data was taken as the averages of the observations falling into relevant phase. Other types of deformation were based on the assumption that economic time progresses with different paces over periods of expansion and contraction. To account for this, an indicator variable can be introduced to reflect the corresponding stage of the cycle. This may be considered as a modification of the approach to account for asymmetry introduced by Neftci (1984).

Recently, the issue of time-deformation for cyclical data is widely discussed in the field of signal processing, speech recognition, biology, etc. As one of the recent developments, Gray and Zhang (1988) studied the data obeying multiplicative group composition law. The multiplicative stationary (M-stationary) continuous Euler process, which is characterized by elongating cyclical behavior, was shown to have a stationary dual as a continuous autoregressive process in logarithmically deformed time domain. Later, Vijverberg and Gray (2004) discretized continuous Euler process into stationary discrete ARMA process. Gray, Vijverburg, and Woodward (2004) developed forecasting and spectral analysis methodology for discrete M-stationary processes using simulated data and bat echolocation signals. Vijverberg (2006) applied the methodology for the study of residential investment growth data. The main limitation of M-stationary processes is that they describe data with elongating cycles, which may be not appropriate for many economic variables.

The time-deformation process used in this paper was developed by Jiang, Gray and Woodward (JGW) (2006). The so-called G(λ)-stationary process is the generalization of different stationarity concepts, such as additive and multiplicative stationarity. This method may be applied to data with frequency changing systematically in time, either increasing or decreasing. The time scale is transformed using the Box-Cox transformation, where λ is the parameter of transformation. Different values of λ correspond to different cyclical behaviors. Non-stationary data in regular time scale is assumed to be G(λ)-stationary and can be converted into the dual process, which is stationary in the deformed time. Estimated parameters can be easily transformed back into the regular time.

In the original paper by Jiang, Gray and Woodward, time deformation is applied to simulated data and geophysical data with the clear cyclical signal embodied. The novel approach in this paper is application of G(λ) transformation to the economic data. While logarithmic time scale deformation (which assumes that data follows M-stationary process) was studied in previous works, fitting G(λ)-stationary process to the economic series is a completely new way to explore data characteristics.

Another new feature in this paper is introducing time deformation in the framework of unobserved component (UC) time series models. Unlike ARIMA models, UC model enables a convenient tool to model stochastic trend, seasonal and cyclical components explicitly. Traditional Kalman filtering and smoothing algorithms allow to obtain likelihood function for parameter estimation within the classical or Bayesian framework. Clark (1987), Harvey (1989), Harvey and Jaeger (1993) developed the general model for structural time series as a linear combination of explicit trend and cycle components. Later, Tripodis and Penzer (2006) reviewed the composition of seasonal component in (UC) models. Koopman, Ooms, and Hindrayanto (2006) discussed issues of estimation, identification and forecasting in the general class of periodic UC models. Different modifications of the the original UC model were considered by Perron and Wada (2005) and Morley, Nelson, and Zivot (2002) in application to the U.S. economic activity. Recent additions to the works on UC modeling of business cycles include Koopman and Lee (2005), who used state space model (SSM) to test the asymmetry in unemployment, private domestic investment, and GDP data. The asymmetry is modeled based on the steepness of the cycle - different frequencies are attached to ascending or descending periods. Varying periodic activity is modeled as the deviation of actual frequency from the underlying basic frequency. Koopman and Wong (2006) adopted time dependent sample spectrum to find out time-varying parameters in the UC model of the U.S. GDP and Industrial Product Index.

UC models provide a useful tool to explore structural data and using it with time deformation tools may provide new information about behavior of many economic periodic variables.

This paper is organized as follows: section 2 considers state space formulation of the UC model. Section 3 discusses the time deformation algorithm proposed. Empirical application to the U.S. GDP and unemployment time series is carried out in section 4, and section 5 concludes.

2. Periodic structural time series model

Following Harvey (1993), the univariate structural time series model can be constructed as follows:

$$y_t = \mu_t + \psi_t + \epsilon_t, \quad t = 1, \ldots, T \tag{2.1}$$

where y_t is the observation at time t, μ_t is trend, Ψ_t is cyclical component, and ϵ_t is idiosyncratic shock with $\epsilon_t \sim NID(0, \sigma_\epsilon^2)$.

Trend component is built as the local level model:

$$\mu_{t+1} = \mu_t + \beta_t + \xi_t, \qquad \xi_t \sim NID(0, \sigma_\xi^2), \tag{2.2}$$

$$\beta_{t+1} = \beta_t + \zeta_t, \qquad \zeta_t \sim NID(0, \sigma_\zeta^2), \tag{2.3}$$

where level and slope innovations, ξ_t and ζ_t, respectively, are normally and independently distributed. Such specification allows to account for different behaviors of the trend component μ_t. If $\sigma_\zeta^2 = 0$, then $\beta_{t+1} = \beta_t = \beta$ and trend follows random walk with drift. If $\sigma_\xi^2 = 0$, trend becomes integrated random walk model. In case both innovations, ξ_t and ζ_t, have zero variance, deterministic trend specification is obtained.

The deterministic cyclical component is specified as follows:

$$\psi_t = a * \cos(\theta_t t - b) \quad a \neq 0, \theta_t \neq 0 \tag{2.4}$$

with a, θ_t, and b being amplitude, frequency, and phase respectively. Frequency, θ_t, is measured in radians with $0 < \theta_t < \pi$. Period of the cycle is obtained as $2\pi / \theta_t$. In case frequency is constant, $\theta_t = \theta$, we obtain a stationary cycle with $\Psi_{(t-\tau)} = \Psi_{(t+\tau)}$. Time-varying frequency allows to model elongating and dampening cycles in the time series. Koopman and Lee (2005) modeled asymmetry of the cycle based on its steepness: $\theta_t = \theta + \gamma \dot{\psi}_t$, where $\dot{\psi}_t = \vartheta \psi_t / \vartheta(\theta t) = -a\sin(\theta t - b)$ is the derivative of the cycle component in (2.4), which determines whether the cycle is ascending or descending. Such approach allows to capture different frequencies over rising or declining periods of a single cycle, but it still assumes that all cycles present in data are identical.

Specification (2.4) can be converted into the autoregressive process (see Appendix A):

$$\begin{pmatrix}\psi_{(t+1)}\\ \dot{\psi}_{(t+1)}\end{pmatrix}=\begin{bmatrix}\cos\theta_t & \sin\theta_t\\ -\sin\theta_t & \cos\theta_t\end{bmatrix}\begin{pmatrix}\psi_t\\ \dot{\psi}_t\end{pmatrix} \tag{2.5}$$

Adding a damping term ρ and cyclical innovations to (2.5) provides stochastic version of cycle:

$$\begin{pmatrix}\psi_{(t+1)}\\ \dot{\psi}_{(t+1)}\end{pmatrix}=\rho\begin{bmatrix}\cos\theta_t & \sin\theta_t\\ -\sin\theta_t & \cos\theta_t\end{bmatrix}\begin{pmatrix}\psi_t\\ \dot{\psi}_t\end{pmatrix}+\begin{pmatrix}k_t\\ \dot{k}_t\end{pmatrix} \tag{2.6}$$

with $|\rho|<1$ and $k_t, \dot{k}_t \sim NID(0,\sigma_k^2)$

All formulas considered above for structural time series can be conveniently put into the linear state-space form with observation and transition equations defined as:

$$y_t=Z_t\alpha_t+\epsilon_t \qquad \epsilon_t\sim NID(0,H_t) \tag{2.7}$$

$$\alpha_{t+1}=T_t\alpha_t+\eta_t \qquad \eta_t\sim NID(0,Q_t) \tag{2.8}$$

with initial state vector $\alpha_1\sim NID(\alpha_1, P_1)$, and disturbances ϵ_t and η_t being mutually and serially uncorrelated Gaussian processes.

The matrices of state-space form are given as:

$$\alpha_t=(\mu_t \quad \beta_t \quad \psi_t \quad \dot{\psi}_t)' \tag{2.9}$$

$$\eta_t=(\xi_t \quad \zeta_t \quad k_t \quad \dot{k}_t)' \tag{2.10}$$

$$Z_t=[1 \quad 0 \quad 1 \quad 0] \tag{2.11}$$

$$T_t=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \rho\cos\theta_t & \rho\sin\theta_t\\ 0 & 0 & -\rho\sin\theta_t & \rho\cos\theta_t\end{bmatrix} \tag{2.12}$$

$$Q_t=\begin{bmatrix}\sigma_\xi^2 & 0 & 0 & 0\\ 0 & \sigma_\zeta^2 & 0 & 0\\ 0 & 0 & \sigma_k^2 & 0\\ 0 & 0 & 0 & \sigma_k^2\end{bmatrix} \qquad (2.13)[1]$$

Estimations of the parameters of the model is implemented via Kalman filtering and smoothing algorithm (Durbin and Koopman (2001, 2002) - see Appendix B).

1 Here the time subscript for matrix Q can actually be omitted since its components are not time varying.

3. Time deformation for nonstationary data

The most common stationarity concept may be expressed as a general group composition law:

$$E[(X(t)-\mu)(X(f(t,\tau))-\mu)] = C_X(\tau) \quad (3.1)$$

Regular stationarity assumes additive structure of $f(t,\tau)$: $E[(X(t)-\mu)(X(t+\tau)-\mu)] = C_X(\tau)$. Gray and Zhang (1988) considered the multiplicative composition law: $E[(X(t)-\mu)(X(t*\tau)-\mu)] = R_X(\tau)$. The latter may be applied for data exhibiting elongating cyclical behavior and is referred to as M-stationary process. Jiang, Gray, and Woodward (2006) develop the G(λ)-stationary process which obeys the following group composition law: $f(t,\tau)=(t^{\lambda}+\tau\lambda)^{1/\lambda}$ with $\lambda\in(-\infty,\infty)$. The following definition from JGW (2006) defines the *G(λ)-s*tationary process.

Definition 1. *Let X(t) be a stochastic process defined* $t\in(0,\infty)$ *such that for any* $(t^{\lambda}+\tau\lambda)\in(0,\infty)$, *and constant* $\lambda\in(-\infty,\infty)$,

i. $E[X(t)] = \mu$,

ii. $var[X(t)] = \sigma^2 < \infty$,

iii. $E[(X(t)-\mu)(X((t^{\lambda}+\tau\lambda)^{1/\lambda})-\mu) = B_X(\tau;\lambda)$. (3.2)

Then X(t) will be called a G(λ)-stationary process.

$B_X(\tau;\lambda)$ is referred to as the G(λ)-autocovariance of $X(t)$ with the following property: $B_X(-\tau;\lambda) = B_X(\tau;\lambda)\ \forall\lambda$.

It can be shown that *G(λ)*-stationarity covers a wide range of different types of stationarity. When $\lambda=1$, $X(t)$ obeys traditional additive stationarity, while $\lambda=0$ leads to M-stationarity.

The *G(λ)*-autocorrelation function of $X(t)$ is defined as $\rho_X(\tau;\lambda)=\dfrac{B_X(\tau;\lambda)}{Var(X(t))}=\dfrac{B_X(\tau;\lambda)}{B_X(0;\lambda)}$. (3.3)

From the definition it is obvious that autocovariance of $X(t)$ depends not only on lag length but also on parameter λ. *G(λ)*-stationary process appears to be non-stationary on the regular time scale. However, it has its stationary dual process $\{Y(u);\ u\in(-\infty,\infty)\}$, which is defined on the time scale over u. Thus, $X(t)=Y(u)$ on $t\in(0,\infty)$.

The transformation functions connecting *t*- and *u*-based scales are:

$$u=g(t)=\frac{t^{\lambda}-1}{\lambda} \quad (3.4)$$

$$t=g'(u)=(u\lambda+1)^{\frac{1}{\lambda}} \quad (3.5)$$

As shown in JGW (2006), $B_X(\tau;\lambda)=C_Y(\tau)$, thus autocovariance of $Y(u)$ depends only on lag length τ, and $Y(u)$ is stationary in the regular sense over the deformed time $u=\frac{t_1^\lambda-1}{\lambda},\frac{t_2^\lambda-1}{\lambda},\frac{t_3^\lambda-1}{\lambda},....,\frac{t_N^\lambda-1}{\lambda}$.

The simplest example of *G(λ)*-stationary function is trigonometric periodic function with constant amplitude and zero phase:

$$X(t)=\sin(2\pi\varphi\frac{t^\lambda-1}{\lambda}) \qquad (3.6)$$

The corresponding dual is $Y(u)=\sin(2\pi\varphi u)$. For the periodic function, parameter λ determines the length of the cycle: $\lambda<1$ generates elongating cycles; $\lambda=1$ generates stationary cycles, while $\lambda>1$ produces fading cycles (see Figure 1).

The typical class of *G(λ)*-stationary process is *G(p,q;λ)* process:

$$\prod_i^p (t^{1-\lambda}D-\alpha_i)X(t)=\prod_j^q (t^{1-\lambda}D-\beta_j)a(t) \qquad (3.7)$$

where $t>0$, D is differential operator, α_i and β_j are constants, p, q = 1,2,3..., and $a(t)$ is *G(λ)*-white noise (see JGW (2006)). If $X(t)$ follows *G(p,q;λ)*-process, it has a stationary dual $Y(u)$:

$$\prod_i^p (D-\alpha_i)Y(u)=\prod_j^q (D-\beta_j)\epsilon(t) \qquad (3.8)$$

which is a continuous *ARMA(p,q)* process. Since $Y(u)$ is stationary in the domain (u), it may be used for estimation with regular econometric tools. However, several technical issues considered below have to be resolved first.

Interpolating equally spaced sample from G(p,q;λ) process

The discrete observed data sample $X(t_k)$ generated in previous step will correspond to discrete Y_k following *ARMA(p,q)* process (3.8). However, Y_k sample will be spaced over unequal intervals due to non-linear transformation (3.4). This may not be suitable for econometric estimation – for example, *ARMA* models assume that data were taken over equally spaced intervals. One way to obtain equally spaced realizations is via interpolation technique, for, example linear interpolation:

$$Z(t,\{Z_k\})=\frac{t_{k+1}-t}{t_{k+1}-t_k}Z_k+\frac{t-t_k}{t_{k+1}-t_k}Z_{k+1} \qquad (3.9)$$

For the convenience of relating two data sets, interpolation may be accomplished for the same number of data points as in the original data. That is, if $t=1..N$, with $\Delta t=1$, we have $u=\frac{t_1^\lambda-1}{\lambda},....,\frac{t_N^\lambda-1}{\lambda}$, and $\Delta u=\frac{t_N^\lambda-t_1^\lambda}{\lambda(N-1)}$. The time step Δu in the equally spaced u-scale is then may be normalized to correspond to the time interval Δt in t-scale, so we could plot two series in the same plot.

Estimating λ and offset Λ. The Q-statistic based on $G(\lambda)$-autocorrelation functions allows to estimate the parameter λ and offset Λ (JGW (2006)):

$$Q_X(\lambda,\Lambda)=\sum_{k=0}^{K}[(\rho_1(k;\lambda,\Lambda)-\rho_{1.5}(k;\lambda,\Lambda))^2+(\rho_2(k;\lambda,\Lambda)-\rho_{1.5}(k;\lambda,\Lambda))^2] \tag{3.10}$$

where k=1,2,..,n, and $\rho_1(k;\lambda,\Lambda)$, $\rho_{1.5}(k;\lambda,\Lambda)$, and $\rho_2(k;\lambda,\Lambda)$ are G(λ)-autocorrelations from subsamples X_{1k}, $X_{1.5k}$, and X_{2k}, respectively. Here X_{1k} and X_{2k} represent two equal halves of the data sample. $X_{1.5k}$ is a subsample consisting of the second half of X_{1k} and first half of X_{2k}. Selecting several values for initial range of λ and Λ gives the statistics (3.10). The true values are those which minimize (3.10).

The offset Λ plays an important role since the data has a finite origin and it's critical to find a $G(\lambda)$-process that fits the sequence of our observations. For example, if actual data are timed as $t=1,2,3,...,N$ and offset is $\Lambda=100$, it corresponds to the $G(\lambda)$-process at time points $101, 102, 103,....,100+N$.

4. Empirical evidence

4.1 Data description

In this section two time series from the US economy will be analyzed: real gross domestic product (GDP) and unemployment (Un). The GDP data is quarterly chain linked series compiled by BEA and unemployment data is monthly percentage unemployment rate reported by BLS. Both data sets are seasonally adjusted at the source and cover the period between 1970 and 2007 consisting of 148 and 444 observations for GDP and unemployment correspondingly. Plots of the data are shown in Figure 2.

4.2 Estimation of the time-scale parameters

Estimated parameter of the time-deformation parameter λ and shift are reported in Table 1. For both series estimated values of λ are smaller than 1, indicating the elongating cyclical nature of the data. For the GDP data λ equals to 0, suggesting that GDP follows pure M-stationary process shifted by 76 time intervals. While for unemployment data the elongating pattern can be supported by visual inspection of the original data (Figure 2.b), it is not so obvious for the GDP. However, the elongating nature of the GDP cycle can be found after we estimate UC model in section 4.3 – plot of the smoothed GDP cycle (Figure 3.b) clearly exhibits periods of increasing length. Increasing timing for the GDP cycle also corresponds to the NBER business cycles chronology for the considered period.

Table 1. Estimated parameters of time-deformation.

	λ	Λ
GDP	0	76
Un	0.7	1

Both series were transformed according to the estimated parameters in Table 1 and then interpolated on equally spaced time scales. To correspond the transformed data to the original data, time interval in the equally-spaced deformed u-scale was normalized to be equal to the regular step in the t-scale. Plots of the original series along with the transformed data are shown in Figure 2. While values of GDP and unemployment remain the same for original and transformed data, the timing is different. At the beginning of the time scale the data is prolonged, while near the ends the data points are suppressed.

4.3 Fitting SSM model to the data.

Unobserved component model (2.7-2.8) was estimated with original and transformed data[2]. Disturbance variances are estimated as log-variances: $\log\sigma_{\epsilon}^2$, $\log\sigma_{\zeta}^2$, $\log\sigma_{k}^2$, as well as frequency parameter $\log\theta$. Dumping term $0<\rho<1$ is transformed as $(1+\exp^{\rho})^{-1}$. For the unemployment series, the trend innovations variance σ_{ς}^2 is kept smooth to avoid too much variability in the trend. Estimation results are reported in Table 2.

Table 2. UC model estimation results (p-values are given in brackets).

	GDP	GDP (transformed)	Un	Un (transformed)
σ_{ϵ}^2	0	0	0	0
σ_{ς}^2	0	0	5e-6	5e-6
σ_{k}^2	5.272e-005 (0.00)	3.803e-005 (0.01)	2.575e-002 (0.00)	1.928e-002 (0.00)
θ	0.194 (0.00)	0.190 (0.00)	0.072 (0.00)	0.073 (0.00)
period	32.38	33.0	87.2	86.0
ρ	0.946 (0.00)	0.963 (0.00)	0.996 (0.00)	0.999 (0.00)
Log-likelihood	491.9	509.9	149.1	211.3
J-B normality test	23.4 (0.00)	4.83 (0.09)	8.0 (0.02)	5.5 (0.06)
L-B (20) test	82.8 (0.00)	45.9 (0.00)	184.1 (0.00)	146.6 (0.00)

These results are consistent with the previous literature. For both series, irregular variance and trend variance appear to be insignificantly different from zero, which is a common feature of many

2 Trend level innovations variance σ_{ξ}^2 was found to be insignificant in this as well as in many preceeding studies, thereby is excluded from the UC specification (2.2).

macroeconomic time series. Estimated variance of the cycle component is found to be smaller for transformed data in both cases. Average cycle length differs slightly between the original and the transformed data, but the difference is not substantial - less than one corresponding time step for both data sets. Transformed data is also marked with an increase in persistence coefficient. Test results report a significant increase in the likelihood values for the estimation over the transformed GDP and unemployment data. Jarque-Bera normality test suggests that normality assumption is clearly to be rejected for the original GDP data and can be accepted at 2% level for the original Un series. For the transformed data, the Null of normality can be accepted at 9% and 6% levels of significance for GDP and Un respectively. Ljung-Box serial correlation test up to twenty lags suggests that all four specifications reject the Null about the absence of serial correlation. However, results for the transformed data show a considerable improvement in reducing the correlation. Plots of the decomposed smoothed trend and cycle components for all series are shown in Figures 3-6. As shown in the plots, cyclical behavior of the original data is subject to periods of increasing length, while deformed data is characterized by approximately equal periodicity.

5. Conclusion.

The goal of this paper is to consider the econometric estimation with covariance non-stationary data. A possible method to treat such data, is the modification of the original time scale into a new scale, in which non-stationarity of data is removed. The methodology developed by Jiang, Gray and Woodward (2006) is employed for non-linear transformation of the original time scale. In the empirical estimation, US GDP and unemployment data were studied; and for both series cyclical component clearly showed elongating behavior. The data was interpolated into the new time domain and used for the estimation in the Unobservable Component model framework. Estimated parameters on the transformed time-scale are consistent with those reported in the literature, while estimation statistics improved.

As a plausible future research, we suggest the exploration of the case where the parameter of time deformation relating two time scales may be non-constant. This would allow to model the data with both elongating and dampening periods simultaneously, which is a more realistic scenario for the economic time series.

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Appendix A. Cyclical component specification.

The deterministic cycle is specified as follows:

$$\psi_t = a * \cos(\theta t - b) \qquad a \neq 0,\ \theta \neq 0 \qquad \text{(A.1)}$$

with $a, \theta,$ and b being amplitude, frequency, and phase respectively[3].

Using the following trigonometric identities:

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(x \pm y) = \cos(x)\sin(y) \pm \sin(x)\cos(y), \qquad \text{(A.2)}$$

(A.1) can be expressed as follows:

$$\psi_t = a * \cos(\theta t)\cos(b) + a\sin(\theta t)\sin(b) \qquad \text{(A.2)}$$

Replacing constant values $a\cos(b)$ and $a\sin(b)$ with α and β respectively, (A.2) becomes:

$$\psi_t = \alpha * \cos(\theta t) + \beta \sin(\theta t) \qquad \text{(A.3)}$$

which has the following partial derivative wrt. to θt :

$$\dot{\psi}_t = -\alpha \sin(\theta t) + \beta \cos(\theta t) \qquad \text{(A.4)}$$

Now, making one-step forward iteration for Ψ_t and $\dot{\Psi}_t$ gives:

$$\psi_{t+1} = \alpha * \cos(\theta(t+1)) + \beta \sin(\theta(t+1)) \qquad \text{(A.5)}$$

$$\dot{\psi}_{t+1} = -\alpha * \sin(\theta(t+1)) + \beta \cos(\theta(t+1)) \qquad \text{(A.5')}$$

Applying (A.2) to (A.5) and (A.5'):

$$\psi_{t+1} = \alpha * \cos(\theta t)\cos(\theta) - \alpha \sin(\theta t)\sin(\theta) + \beta \cos(\theta t)\sin(\theta) + \beta \sin(\theta t)\cos(\theta) =$$

$$= \cos(\theta) * \psi_t + \sin(\theta) * \dot{\psi}_t \qquad \text{(A.6)}$$

$$\dot{\psi}_{t+1} = -\alpha * \cos(\theta t)\sin(\theta) - \alpha \sin(\theta t)\cos(\theta) + \beta \cos(\theta t)\cos(\theta) + \beta \sin(\theta t)\sin(\theta) =$$

$$= -\sin(\theta) * \psi_t + \cos(\theta) * \dot{\psi}_t \qquad \text{(A.6')}$$

Or, in matrix form:

$$\begin{pmatrix} \Psi_{(t+1)} \\ \dot{\Psi}_{(t+1)} \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \Psi_t \\ \dot{\psi}_t \end{pmatrix} \qquad \text{(A.7)}$$

3 Time subscript on frequency parameter θ is omitted for the simplicity of expostition.

Appendix B. UC Modeling and Kalman filter.

A linear Gaussian state space model is defined by observation and state equations:

$$y_t = Z_t \alpha_t + \epsilon_t \qquad \varepsilon_t \sim NID(0, G_t) \tag{B.1}$$

$$\alpha_{(t+1)} = H_t \alpha_t + \eta_t \qquad \eta_t \sim NID(0, Q_t) \tag{B.2}$$

with an initial state vector $\alpha_1 \sim (a_1, P_1)$ and disturbances ε_t and η_t being mutually and serially uncorrelated Gaussian processes.

Kalman filter represents a recursive algorithm which estimates the state of the process α_t based on the past observations $y_1 \ldots y_{t-1}$ via the minimization of the mean squared error. In addition to the estimated states, it can provide a likelihood function used for the parameter estimation.

Assuming the initial state (α_t, P_1) is known, the following estimates are computed as follows:

$$v_t = y_t - Z_t a_t \tag{B.3}$$

$$F_t = Z_t P_t Z_t' + G_t \tag{B.4}$$

$$K_t = H_t P_t Z_t' F_t^{-1} \tag{B.5}$$

$$a_{(t+1)} = H_t a_t + K_t v_t \tag{B.6}$$

$$P_{(t+1)} = H_t P_t H_t' + R_t Q_t R_t' - K_t - K_t F_t K_t' \tag{B.7}$$

where:

$$a_t = E(\alpha_t | y_1 \ldots y_{(t-1)}) \tag{B.8}$$

$$P_t = Var(\alpha_t | y_1 \ldots y_{(t-1)}) \tag{B.9}$$

The likelihood function then is:

$$\log L(.) = -\frac{(np)}{(2)} \log(2\pi) - \frac{1}{2} \sum_{(t=1)}^{T} (\log |F_t| + v_t' F_t^{(-1)} v_t) \tag{B.10}$$

Kalman smoother estimates α_t based on the whole set of observations $y_1 \ldots y_T$ using the backward recursion:

$$L_t = H_t - K_t Z_t \tag{B.11}$$

$$r_{(t-1)} = Z_t' F_t^{(-1)} v_t + L_t' r_t \tag{B.12}$$

$$N_{(t-1)} = Z_t' F_t^{(-1)} Z_t + L_t' N_t L_t \tag{B.13}$$

$$\hat{\alpha}_t = a_t + P_t r_{(t-1)} \tag{B.14}$$

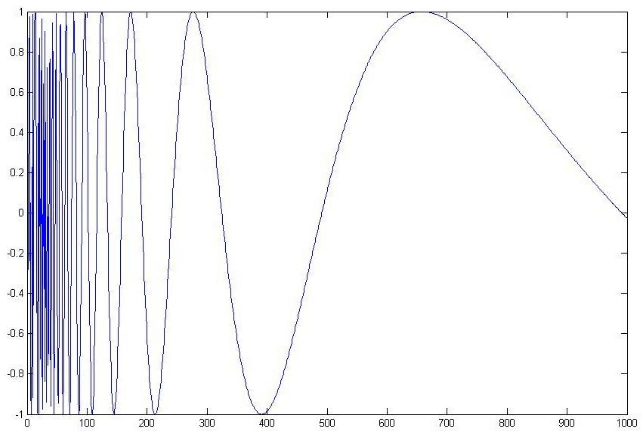
$$V_t = P_t - P_t N_{(t-1)} P_t \tag{B.15}$$

with smoothed state $\hat{\alpha}_t = E(\alpha_t | y)$ having the smallest variance $V_t = Var(\alpha_t | y)$

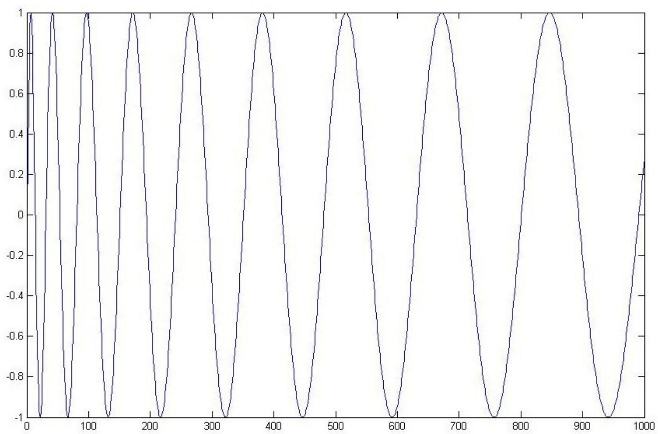
Methods for drawing states α_t based on the whole sample y are well developed in the literature and are easy to implement (Durbin and Koopman 2001, 2002).

Appendix C. Figures.

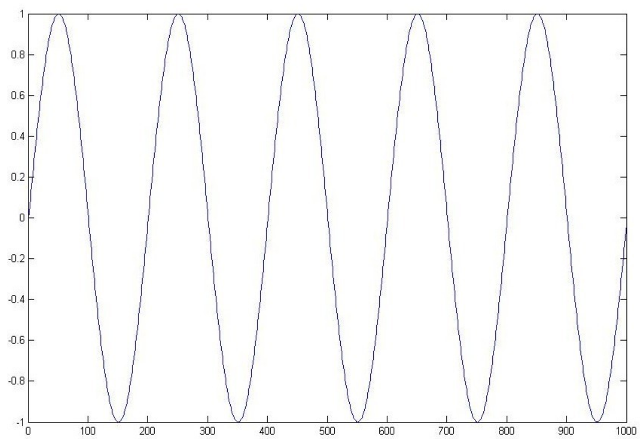
Figure 1. Cycle length for different values of λ.



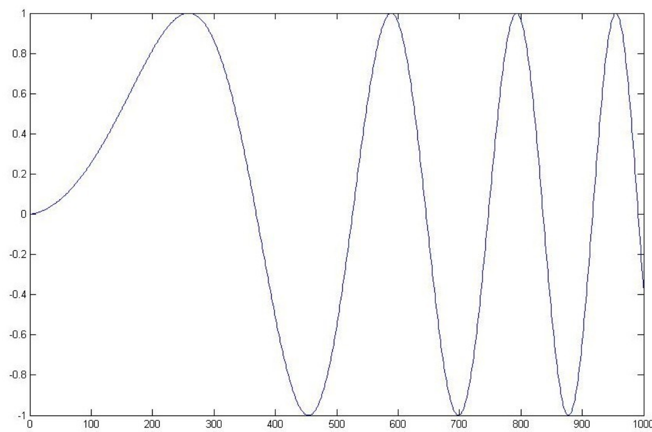
λ=-1



λ=0.5



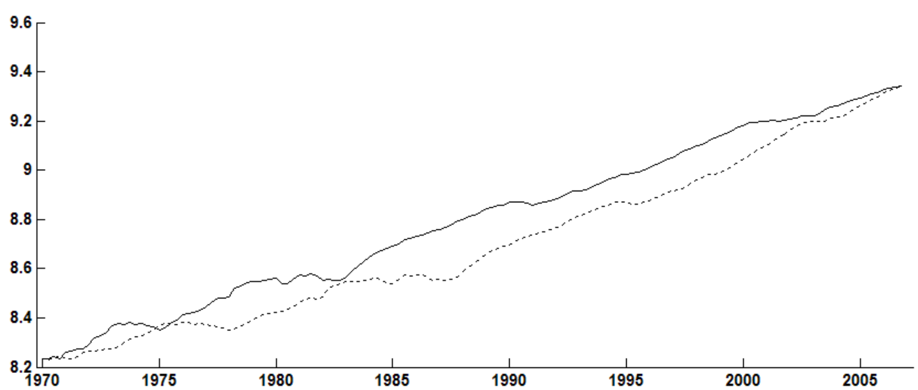
λ=1



λ=2

Figure 2. (a) Original (solid line) and transformed (dashed line) log GDP data. (b) Original (solid line) and transformed (dashed line) unemployment data

(a)



(b)

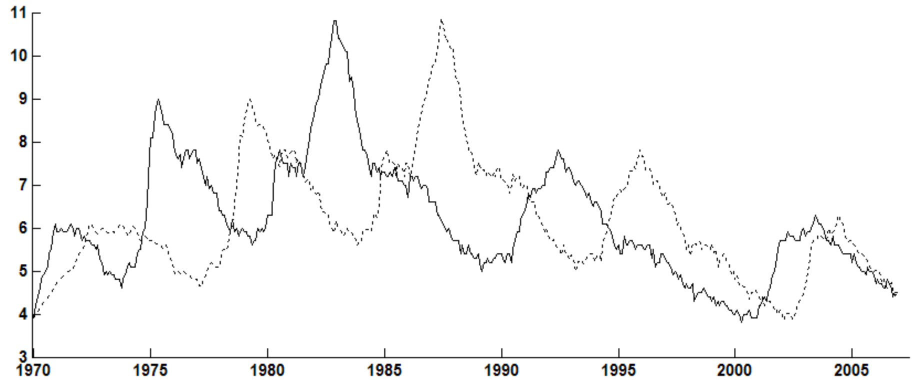
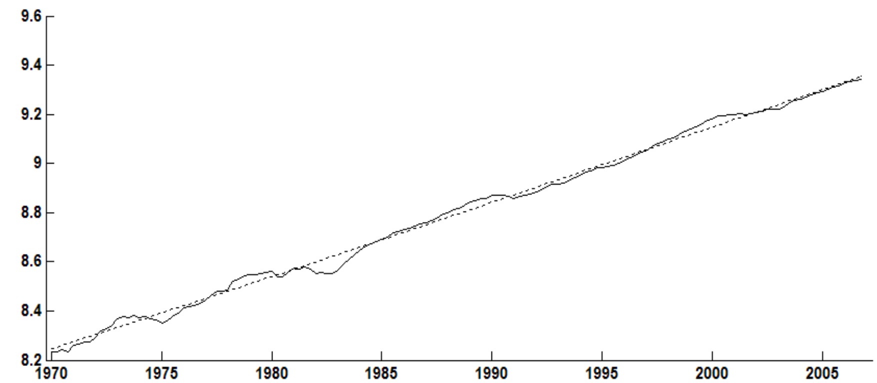


Figure 3. Trend-cycle decomposition of the original GDP data:
(a) The data and smoothed trend. (b) Smoothed cycle.

(a)



(b)

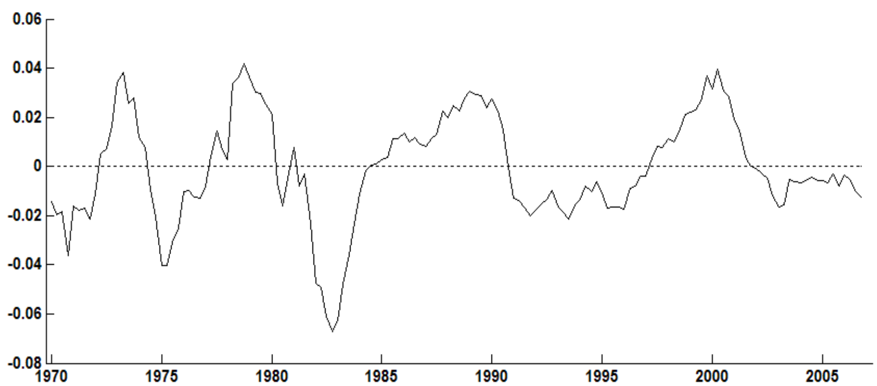
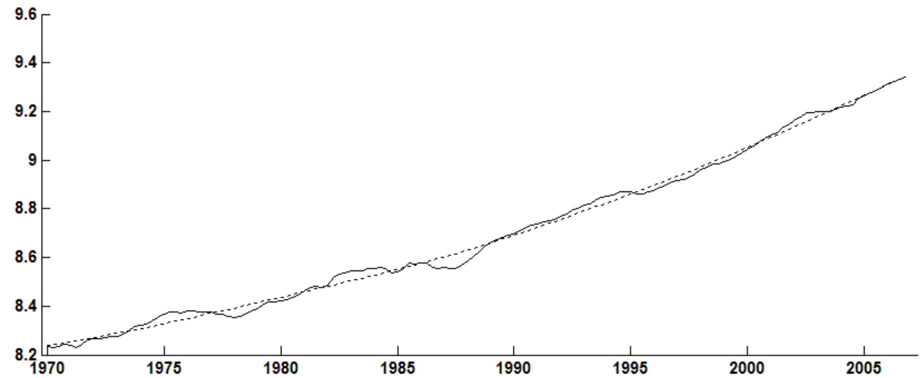


Figure 4. Trend-cycle decomposition of the transformed GDP data:

(a) The data and smoothed trend. (b) Smoothed cycle.

(a)



(b)

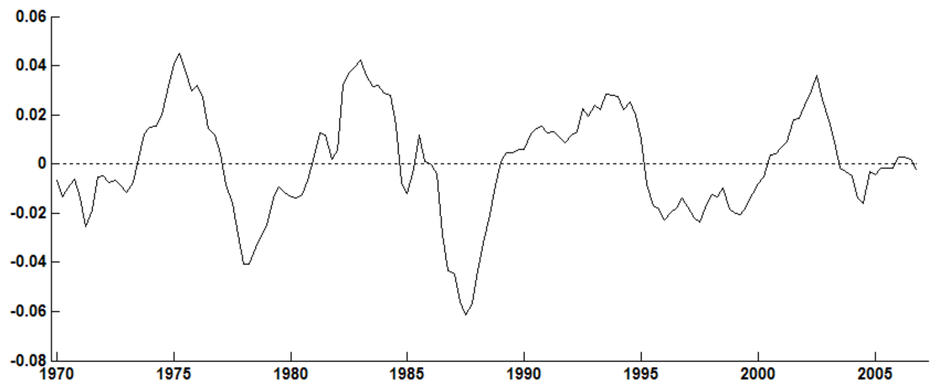
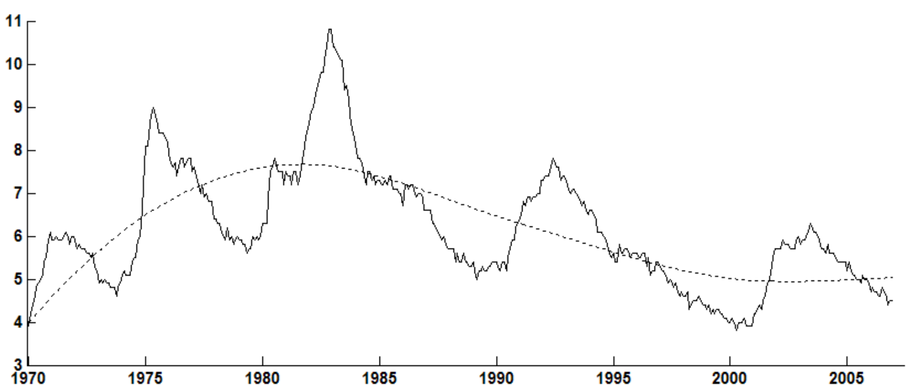


Figure 5. Trend-cycle decomposition of the original unemployment data:

(a) The data and smoothed trend. (b) Smoothed cycle.

(a)



(b)

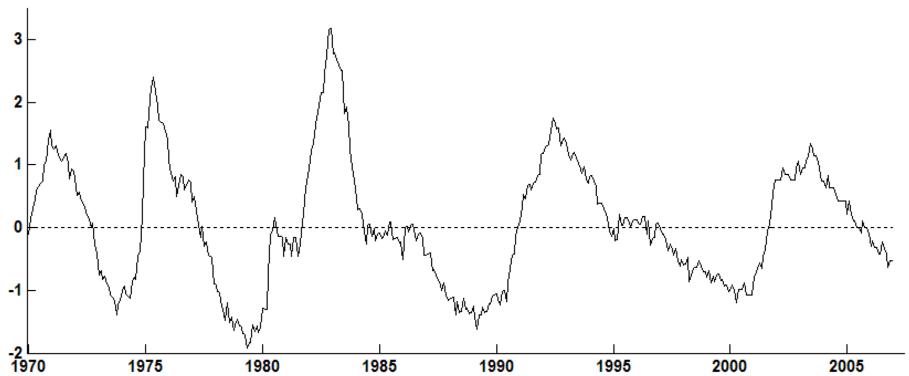
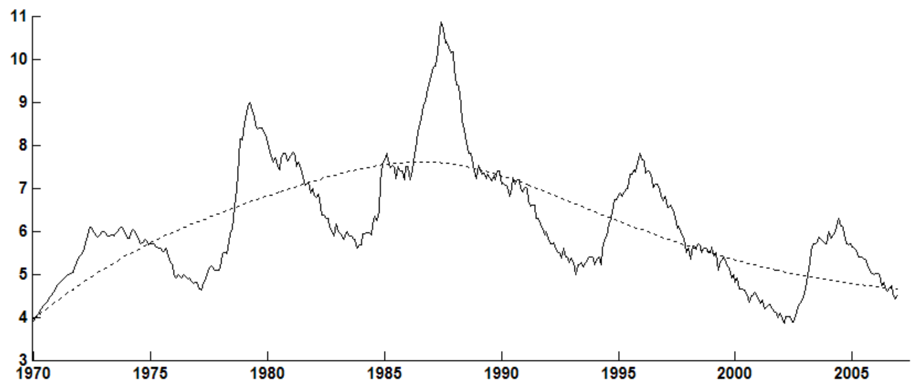


Figure 6. Trend-cycle decomposition of the transformed unemployment data:
(a) The data and smoothed trend. (b) Smoothed cycle.

(a)



(b)