TIME-VARYING RISK PREMIUM IN LARGE CROSS-SECTIONAL EQUITY DATASETS

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Abstract

We develop an econometric methodology to infer the path of risk premia from large unbalanced panel of individual stock returns. We estimate the time-varying risk premia implied by conditional linear asset pricing models where the conditioning includes instruments common to all assets and asset specific instruments. The estimator uses simple weighted two-pass cross-sectional regressions, and we show its consistency and asymptotic normality under increasing cross-sectional and time series dimensions. We address consistent estimation of the asymptotic variance, and testing for asset pricing restrictions induced by the no-arbitrage assumption in large economies. The empirical illustration on returns for about ten thousands US stocks from July 1964 to December 2009 shows that conditional risk premia are large and volatile in crisis periods. They exhibit large positive and negative strays from standard unconditional estimates and follow the macroeconomic cycles. The asset pricing restrictions are rejected for the usual unconditional four-factor model capturing market, size, value and momentum effects.

JEL Classification: C12, C13, C23, C51, C52 , G12.

Keywords: large panel, factor model, risk premium, asset pricing.

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1 Introduction

Risk premia measure financial compensation asked by investors for bearing risk. Risk is influenced by financial and macroeconomic variables. Conditional linear factor models aim at capturing their time-varying influence in a simple setting (see e.g. Shanken (1990), Cochrane (1996), Ferson and Schadt (1996), Ferson and Harvey (1991, 1999), Lettau and Ludvigson (2001), Petkova and Zhang (2005)). Time variation in risk is known to bias unconditional estimates of alphas and betas, and therefore asset pricing test conclusions (Jagannathan and Wang (1996), Lewellen and Nagel (2006), Boguth, Carlson, Fisher and Simutin (2010)). Ghysels (1998) discusses the pros and cons of modeling time-varying betas.

The workhorse to estimate equity risk premia in a linear multi-factor setting is the two-pass cross-sectional regression method developed by Black, Jensen and Scholes (1972) and Fama and MacBeth (1973). Its large and finite sample properties for unconditional linear factor models have been addressed in a series of papers, see e.g. Shanken (1985, 1992), Jagannathan and Wang (1998), Shanken and Zhou (2007), Kan, Robotti and Shanken (2009), and the review paper of Jagannathan, Skoulakis and Wang (2009). Statistical inference for equity risk premia in conditional linear factor model has not yet been formally addressed in the literature despite its empirical relevance.

In this paper we study how we can infer the time-varying behaviour of equity risk premia from large stock return databases by using conditional linear factor models. Our approach is inspired by the recent trend in macro-econometrics and forecasting methods trying to extract cross-sectional and time-series information simultaneously from large panels (see e.g. Stock and Watson (2002a,b), Bai (2003, 2009), Bai and Ng (2002, 2006), Forni, Hallin, Lippi and Reichlin (2000, 2004, 2005), Pesaran (2006)). Ludvigson and Ng (2007, 2009) show that it is a promising route to follow to study bond risk premia. Connor, Hagmann, and Linton (2011) show that large cross-section helps to exploit data more efficiently in a semiparametric characteristic-based factor model of stock returns. It is also inspired by the framework underlying the Arbitrage Pricing Theory (APT). Approximate factor structures with nondiagonal error covariance matrices (Chamberlain and Rothschild (1983, CR)) address the potential empirical mismatch of exact factor structures with diagonal error covariance matrices underlying the original APT of Ross (1976). Under weak cross-sectional dependence among error terms, they generate no-arbitrage restrictions in large economies where the number of assets grows to infinity. Our paper develops an econometric methodology tailored to the APT framework. We let the number of assets grow to infinity mimicking the large economies of financial theory.

Our approach is further motivated by the potential loss of information and bias induced by grouping stocks to build portfolios in asset pricing tests (Litzenberger and Ramaswamy (1979), Lo and MacKinlay (1990), Berk (2000), Conrad, Cooper and Kaul (2003), Phalippou (2007)). Avramov and Chordia (2006) have already shown that empirical findings given by conditional factor models about anomalies differ a lot when considering single securities instead of portfolios. Ang, Liu and Schwarz (2008) argue that a lot of efficiency may be lost when only considering portfolios as base assets, instead of individual stocks, to estimate equity risk premia in unconditional models. In our approach the large cross-section of stock returns also helps to get accurate estimation of the equity risk premia even if we get noisy time-series estimates of the factor loadings (the betas). Besides, when running asset-pricing tests, Lewellen, Nagel and Shanken (2010) advocate working with a large number of assets instead of working with a small number of portfolios exhibiting a tight factor structure. The former gives us a higher hurdle to meet in judging model explanation based on cross-sectional R^2.

Our theoretical contributions are threefold. First we derive no-arbitrage restrictions in a multi-period economy (Hansen and Richard (1987)) with a continuum of assets and an approximate factor structure (Chamberlain and Rothschild (1983)). We explicitly show the relationship between the ruling out of asymptotic arbitrage opportunities and a testable restriction for large economies in a conditional setting. We also formalize the sampling scheme when observed assets are random draws from an underlying population (Andrews (2005)). Second we derive a new weighted two-pass cross-sectional estimator of the path over time of the risk premia from large unbalanced panels of excess returns. We study its large sample properties in conditional linear factor models where the conditioning includes instruments common to all assets and asset specific instruments. The factor modeling permits conditional heteroskedasticity and cross-sectional dependence in the error terms (see Petersen (2008) for stressing the importance of residual dependence when computing standard errors in finance panel data). We derive consistency and asymptotic normality of our estimates by letting the time dimension T and the cross-section dimension n grow to infinity simultaneously, and not sequentially. We relate the results to bias-corrected estimation (Hahn and Kuersteiner (2002), Hahn and Newey (2004)) accounting for the well-known incidental parameter problem of the panel literature (Neyman and Scott (1948)). We derive all properties for unbalanced panels to avoid the survivorship bias inherent to studies restricted to balanced subsets of available stock return databases (Brown, Goetzmann, Ross (1995)). The two-pass regression approach is simple and particularly easy to implement in an unbalanced setting. This explains our choice over more efficient, but numerically intractable, one-pass ML/GMM estimators or generalized least-squares estimators. When n is of the order of a couple of thousands assets, numerical optimization on a large parameter set or numerical inversion of a large weighting matrix is too challenging and unstable to benefit in practice from the theoretical efficiency gains, unless imposing strong ad hoc structural restrictions. Third we provide a goodness-of-fit test for the conditional factor model underlying the estimation. The test exploits the asymptotic distribution of a weighted sum of squared residuals of the second-pass cross-sectional regression (see Lewellen, Nagel and Shanken (2010), Kan, Robotti and Shanken (2009) for a related approach in unconditional models and asymptotics with fixed n). The construction of the test statistic relies on consistent estimation of large-dimensional sparse covariance matrices by thresholding (Bickel and Levina (2008), El Karoui (2008), Fan, Liao, and Mincheva (2011)). As a by-product, our approach permits inference for the cost of equity on individual stocks, in a time-varying setting (Fama and French (1997)). As known from standard textbooks in corporate finance, the cost of equity is such that cost of equity = risk free rate + factor loadings $\times$ factor risk premia. It is part of the cost of capital and is a central piece for evaluating investment projects by company managers. For pedagogical purposes the three theoretical contributions are first presented in an unconditional setting before being extended to a conditional setting.

For our empirical contributions, we consider the Center for Research in Security Prices (CRSP) database and take the Compustat database to match firm characteristics. The merged dataset comprises about ten thousands stocks with monthly returns from July 1964 to December 2009. We look at factor models popular in the empirical finance literature to explain monthly equity returns. They differ by the choice of the factors. The first model is the CAPM (Sharpe (1964), Lintner (1965)) using market return as the single factor. Then, we consider the three-factor model of Fama and French (1993) based on two additional factors capturing the book-to-market and size effects, and a four-factor extension including a momentum factor (Jegadeesh and Titman (1993), Carhart (1997)). We study both unconditional and conditional factor models (Ferson and Schadt (1996), and Ferson and Harvey (1999)). For the conditional versions we use both macrovariables and firm characteristics as instruments. The estimated path shows that the risk premia are large and volatile in crisis periods, e.g., the oil crisis in 1973-1974, the market crash in October 1987, and the crisis of the recent years. Furthermore, the conditional estimates exhibit large positive and negative strays from standard unconditional estimates and follow the macroeconomic cycles. The asset pricing restrictions are rejected for the usual unconditional four-factor model capturing market, size, value and momentum effects.

The outline of the paper is as follows. In Section 2 we present our approach in an unconditional linear factor setting. In Section 3 we extend all results to cover a conditional linear factor model where the instruments inducing time varying coefficients can be common to all stocks or stock specific. Section 4 contains the empirical results. Section 5 contains the simulation results. Finally, Section 6 concludes. In the Appendix, we gather the technical assumptions and some proofs. We place all omitted proofs in the online supplementary materials. We use high-level assumptions to get our results and show in Appendix 4 that they are all met under a block cross-sectional dependence structure on the error terms in a serially i.i.d. framework.

2 Unconditional factor model

In this section we consider an unconditional linear factor model in order to illustrate the main contributions of the article in a simple setting. This covers the CAPM where the single factor is the excess market return.

2.1 Excess return generation and asset pricing restrictions

We start by describing how excess returns are generated before examining the implications of absence of arbitrage opportunities in terms of restrictions on the return generating process. We combine the constructions of Hansen and Richard (1987) and Andrews (2005) to define a multi-period economy with a continuum of assets having strictly stationary and ergodic return processes. We use such a formal construction to guarantee that (i) the economy is invariant to time shifts, so that we can establish all properties by working at $t = 1$, (ii) time series averages converge almost surely to population expectations, (iii) under a sampling mechanism (see the next section) cross-sectional limits exist and are invariant to reordering of the assets, and (iv) the derived no-arbitrage restriction is empirically testable.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The random vector f admitting values in $\mathbb{R}^K$, and the collection of random variables $\varepsilon(\gamma)$, $\gamma \in [0,1]$, are defined on this probability space. Moreover, let $\beta = (a, b')'$ be a vector function defined on $[0,1]$ with values in $\mathbb{R} \times \mathbb{R}^K$. The dynamics is described by the measurable time-shift transformation S mapping Ω into itself. If $\omega \in \Omega$ is the state of the world at time 0, then $S^t(\omega)$ is the state at time t, where S^t denotes the transformation S applied t times successively. Transformation S is assumed to be measure-preserving and ergodic (i.e., any set in $\mathcal{F}$ invariant under S has measure either 1, or 0).

Assumption APR.1 *The excess returns $R_t(\gamma)$ of asset $\gamma \in [0,1]$ at date $t = 1, 2, ...$ satisfy the unconditional linear factor model:*

$$R_t(\gamma) = a(\gamma) + b(\gamma)' f_t + \varepsilon_t(\gamma), \tag{1}$$

where the random variables $\varepsilon_t(\gamma)$ and f_t are defined by $\varepsilon_t(\gamma, \omega) = \varepsilon[\gamma, S^t(\omega)]$ and $f_t(\omega) = f[S^t(\omega)]$.

Assumption APR.1 defines the excess return processes for an economy with a continuum of assets. The index set is the interval $[0,1]$ without loss of generality. Vector f_t gathers the values of the K observable factors at date t, while the intercept $a(\gamma)$ and factor sensitivities $b(\gamma)$ of asset $\gamma \in [0,1]$ are time invariant. Since transformation S is measure-preserving and ergodic, all processes are strictly stationary and ergodic (Doob (1953)). Let further define $x_t = (1, f_t')'$ which yields the compact formulation:

$$R_t(\gamma) = \beta(\gamma)' x_t + \varepsilon_t(\gamma). \tag{2}$$

In order to define the information sets, let $\mathcal{F}_0 \subset \mathcal{F}$ be a sub sigma-field. Random vector f is assumed measurable w.r.t. $\mathcal{F}_0$. Define $\mathcal{F}_t = \{S^{-t}(A), A \in \mathcal{F}_0\}$, $t = 1, 2, ...$, and assume that $\mathcal{F}_1$ contains $\mathcal{F}_0$. Then, the filtration $\mathcal{F}_t$, $t = 1, 2, ...$, characterizes the information available to investors.

Let us now introduce supplementary assumptions on factors, factor loadings and error terms.

Assumption APR.2 *The matrix $\int b(\gamma) b(\gamma)' d\gamma$ is positive definite.*

Assumption APR.2 implies non-degeneracy in the factor loadings across assets.

Assumption APR.3 *For any $\gamma \in [0,1]$: $E[\varepsilon_t(\gamma) | \mathcal{F}_{t-1}] = 0$ and $Cov[\varepsilon_t(\gamma), f_t | \mathcal{F}_{t-1}] = 0$.*

Hence, the error terms have mean zero and are uncorrelated with the factors conditionally on information $\mathcal{F}_{t-1}$. In Assumption APR.4 (i) below, we impose an approximate factor structure for the conditional distribution of the error terms given $\mathcal{F}_{t-1}$ in almost any countable collection of assets. More precisely, for any sequence (γ_i) in $[0, 1]$, let $\Sigma_{\varepsilon,t,n}$ denote the $n \times n$ conditional variance-covariance matrix of the error vector $[\varepsilon_t(\gamma_1), ..., \varepsilon_t(\gamma_n)]'$ given $\mathcal{F}_{t-1}$, for $n \in \mathbb{N}$. Let μ_Γ be the measure on the set $\Gamma = [0, 1]^{\mathbb{N}}$ of sequences (γ_i) in $[0, 1]$ induced by i.i.d. random sampling from a continuous distribution G with support $[0, 1]$.

Assumption APR.4 *For any sequence (γ_i) in $\mathcal{J}$: (i) $eig_{\max}(\Sigma_{\varepsilon,t,n}) = o(n)$, as $n \to \infty$, P-a.s., (ii) $\inf_{n\geq 1} eig_{\min}(\Sigma_{\varepsilon,t,n}) > 0$, P-a.s., where $\mathcal{J} \subset \Gamma$ is such that $\mu_\Gamma(\mathcal{J}) = 1$, and $eig_{\min}(\Sigma_{\varepsilon,t,n})$ and $eig_{\max}(\Sigma_{\varepsilon,t,n})$ denote the smallest and the largest eigenvalues of matrix $\Sigma_{\varepsilon,t,n}$, (iii) $eig_{\min}(V[f_t|\mathcal{F}_{t-1}]) > 0$, P-a.s.*

Assumption APR.4 (i) is weaker than boundedness of the largest eigenvalue, i.e., $\sup_{n\geq 1} eig_{\max}(\Sigma_{\varepsilon,t,n}) < \infty$, P-a.s., as in CR. This is useful for the checks of Appendix 4 under the block cross-sectional dependence structure. Assumptions APR.4 (ii)-(iii) are mild regularity conditions used in the proof of Proposition 1.

Absence of asymptotic arbitrage opportunities generates asset pricing restrictions in large economies (Ross (1976), CR). We define asymptotic arbitrage opportunities in terms of sequences of portfolios p_n, $n \in \mathbb{N}$. Portfolio p_n is defined by the share $\alpha_{0,n}$ invested in the riskfree asset and the shares $\alpha_{i,n}$ invested in the selected risky assets γ_i for $i = 1, ..., n$. The shares are measurable w.r.t. $\mathcal{F}_0$. Then $C(p_n) = \sum_{i=0}^{n} \alpha_{i,n}$ is the portfolio cost at $t = 0$, and $p_n = C(p_n)R_0 + \sum_{i=1}^{n} \alpha_{i,n} R_1(\gamma_i)$ is the portfolio payoff at $t = 1$, where R_0 denotes the riskfree gross return measurable w.r.t. $\mathcal{F}_0$. We can work with $t = 1$ because of stationarity.

Assumption APR.5 *There are no asymptotic arbitrage opportunities in the economy, that is, there exists no portfolio sequence (p_n) such that $\lim_{n\to\infty} P[p_n \geq 0] = 1$ and $\lim_{n\to\infty} P[C(p_n) \leq 0, p_n > 0] > 0$.*

Assumption APR.5 excludes portfolios that approximate arbitrage opportunities when the number of included assets increases. Arbitrage opportunities are investments with non-positive cost and non-negative payoff in each state of the world, and positive payoff in some states of the world (Hansen and Richard (1987), Definition 2.4). Then, the asset pricing restriction is given in the next Proposition 1.

Proposition 1 *Under Assumptions APR.1-APR.5, there exists a unique vector $\nu \in \mathbb{R}^K$ such that:*

$$a(\gamma) = b(\gamma)'\nu, \tag{3}$$

for almost all $\gamma \in [0, 1]$.

The asset pricing restriction in Proposition 1 can be rewritten as

$$E\left[R_t(\gamma)\right] = b(\gamma)'\lambda, \tag{4}$$

for almost all $\gamma \in [0, 1]$, where $\lambda = \nu + E\left[f_t\right]$ is the vector of the risk premia. In the CAPM, we have $K = 1$ and $\nu = 0$. When a factor $f_{k,t}$ is a portfolio excess return, we also have $\nu_k = 0$, $k = 1, ..., K$.

Proposition 1 differs from CR Theorem 3 in terms of the returns generating framework, the definition of asymptotic arbitrage opportunities, and the derived asset pricing restriction. Specifically, we consider a multi-period economy with conditional information as opposed to a single period unconditional economy as in CR. Such a setting can be easily extended to time varying risk premia in Section 3. We prefer the definition underlying Assumption APR.5 since it corresponds to the definition of arbitrage that is standard in dynamic asset pricing theory (e.g., Duffie (2001)). As pointed out by Hansen and Richard (1987), Ross (1978) has already chosen that type of definition. It also eases the proof. However, in Appendix 2, we derive the link between the no-arbitrage conditions in Assumptions A.1 i) and ii) of CR, written P-a.s. w.r.t. the conditional information $\mathcal{F}_0$ and for almost every countable collection of assets, and the asset pricing restriction (3) valid for the continuum of assets. Hence, we are able to characterize the functions $\beta = (a, b')'$ defined on $[0, 1]$ that are compatible with absence of asymptotic arbitrage opportunities under both definitions of arbitrage in the continuum economy. CR derive the pricing restriction $\sum_{i=1}^{\infty} \left(a(\gamma_i) - b(\gamma_i)'\nu\right)^2 < \infty$, for some $\nu \in \mathbb{R}^K$ and for a given sequence (γ_i), while we derive the restriction (3), for almost all γ. In Appendix 2, we show that the set of sequences (γ_i) such that $\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} \left(a(\gamma_i) - b(\gamma_i)'\nu\right)^2 < \infty$ has measure 1 under μ_Γ, when the asset pricing restriction (3) holds, and measure 0, otherwise. This result is a consequence of the Kolmogorov zero-one law (see e.g. Billingsley (1995)). From the proofs in Appendix 2, it is also seen that, when the asset pricing restriction (3) does not hold, asymptotic arbitrage in the sense of Assumption APR.5, or of Assumptions A.1 i) and ii) of CR, exists for μ_Γ-almost any countable collection of assets. The restriction in Proposition 1 is testable with large equity datasets and large sample sizes (Section 2.5), and therefore is not affected by the Shanken (1982) critique. The next section describes how we get the data from sampling the continuum of assets.

2.2 The sampling scheme

We estimate the risk premia from a sample of observations on returns and factors for n assets and T dates. In available databases, asset returns are not observed for all firms at all dates. We account for the unbalanced nature of the panel through a collection of indicator variables $I(\gamma)$, $\gamma \in [0,1]$, and define $I_t(\gamma,\omega) = I[\gamma, S^t(\omega)]$. Then $I_t(\gamma) = 1$ if the return of asset γ is observable by the econometrician at date t, and 0 otherwise (Connor and Korajczyk (1987)). To ease exposition and to keep the factor structure linear, we assume a missing-at-random design (Rubin (1976), Heckman (1979)), that is, independence between unobservability and returns generation.

Assumption SC.1 *The random variables $I_t(\gamma)$, $\gamma \in [0,1]$, are independent of $\varepsilon_t(\gamma)$, $\gamma \in [0,1]$, and f_t.*

Another design would require an explicit modeling of the link between the unobservability mechanism and the continuum of assets; this would yield a nonlinear factor structure.

Assets are randomly drawn from the population according to a probability distribution G on $[0,1]$. We use a single distribution G in order to avoid the notational burden when working with different distributions on different subintervals of $[0,1]$.

Assumption SC.2 *The random variables γ_i, $i = 1,...,n$, are i.i.d. indices, independent of $\varepsilon_t(\gamma)$, $I_t(\gamma)$, $\gamma \in [0,1]$ and f_t, each with continuous distribution G with support $[0,1]$.*

For any $n,T \in \mathbb{N}$, the excess returns are $R_{i,t} = R_t(\gamma_i)$ and the observability indicators are $I_{i,t} = I_t(\gamma_i)$, for $i = 1,...,n$, and $t = 1,...,T$. The excess return $R_{i,t}$ is observed if and only if $I_{i,t} = 1$. Similarly, let $\beta_i = \beta(\gamma_i) = (a_i, b_i')'$ be the characteristics, $\varepsilon_{i,t} = \varepsilon_t(\gamma_i)$ the error terms and $\sigma_{ij,t} = E[\varepsilon_{i,t}\varepsilon_{j,t}|x_{\underline{t}}, \gamma_i, \gamma_j]$ the conditional variances and covariances of the assets in the sample, where $x_{\underline{t}} = \{x_t, x_{t-1}, ...\}$. By random sampling, we get a random coefficient panel model (e.g. Wooldridge (2002)). The characteristic β_i of asset i is random, and potentially correlated with the error terms $\varepsilon_{i,t}$ and the observability indicators $I_{i,t}$, as well as the conditional variances $\sigma_{ii,t}$, through the index γ_i. If the a_is and b_is were treated as deterministic, and not as realizations of random variables, invoking cross-sectional LLNs and CLTs as in some assumptions and parts of the proofs would have no sense. Moreover, cross-sectional limits would be dependent on the selected ordering of the assets. Instead, our assumptions and results do not rely on a specific ordering of assets. Random elements $(\beta_i', \sigma_{ii,t}, \varepsilon_{i,t}, I_{i,t})'$, $i = 1, ..., n$, are exchangeable (Andrews (2005)). Hence, assets randomly drawn from the population have ex-ante the same features. However, given a specific realization of the indices in the sample, assets have ex-post heterogeneous features.

2.3 Asymptotic properties of risk premium estimation

We consider a two-pass approach (Fama and MacBeth (1973), Black, Jensen and Scholes (1972)) building on Equations (1) and (3).

First Pass: The first pass consists in computing time-series OLS estimators $\hat{\beta}_i = (\hat{a}_i, \hat{b}_i')' = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_t I_{i,t} x_t R_{i,t}$, for $i = 1, ..., n$, where $\hat{Q}_{x,i} = \frac{1}{T_i} \sum_t I_{i,t} x_t x_t'$ and $T_i = \sum_t I_{i,t}$. In available panels the random sample size T_i for asset i can be small, and the inversion of matrix $\hat{Q}_{x,i}$ can be numerically unstable. This can yield unreliable estimates of β_i. To address this, we introduce a trimming device: $\mathbf{1}_i^{\chi} = \mathbf{1}\left\{CN\left(\hat{Q}_{x,i}\right) \leq \chi_{1,T}, \tau_{i,T} \leq \chi_{2,T}\right\}$, where $CN\left(\hat{Q}_{x,i}\right) = \sqrt{eig_{\max}\left(\hat{Q}_{x,i}\right)/eig_{\min}\left(\hat{Q}_{x,i}\right)}$ denotes the condition number of matrix $\hat{Q}_{x,i}$, $\tau_{i,T} = T/T_i$, and the two sequences $\chi_{1,T} > 0$ and $\chi_{2,T} > 0$ diverge asymptotically. The first trimming condition $\{CN\left(\hat{Q}_{x,i}\right) \leq \chi_{1,T}\}$ keeps in the cross-section only assets for which the time series regression is not too badly conditioned. A too large value of $CN\left(\hat{Q}_{x,i}\right) = 1/CN\left(\hat{Q}_{x,i}^{-1}\right)$ indicates multicollinearity problems and ill-conditioning (Belsley, Kuh, and Welsch (2004), Greene (2008)). The second trimming condition $\{\tau_{i,T} \leq \chi_{2,T}\}$ keeps in the cross-section only assets for which the time series is not too short.

Second Pass: The second pass consists in computing a cross-sectional estimator of ν by regressing the $\hat{a}_i$'s on the $\hat{b}_i$'s keeping the non-trimmed assets only. We use a WLS approach. The weights are estimates of $w_i = v_i^{-1}$, where the v_i are the asymptotic variances of the standardized errors $\sqrt{T}\left(\hat{a}_i - \hat{b}_i'\nu\right)$ in the cross-sectional regression for large T. We have $v_i = \tau_i c_\nu' Q_x^{-1} S_{ii} Q_x^{-1} c_\nu$, where $Q_x = E\left[x_t x_t'\right]$, $S_{ii} = \operatorname*{plim}_{T\to\infty} \frac{1}{T} \sum_t \sigma_{ii,t} x_t x_t' = E\left[\varepsilon_{i,t}^2 x_t x_t' | \gamma_i\right]$, $\tau_i = \operatorname*{plim}_{T\to\infty} \tau_{i,T} = E[I_{i,t}|\gamma_i]^{-1}$, and $c_\nu = (1, -\nu')'$. We use the estimates $\hat{v}_i = \tau_{i,T} c'_{\hat{\nu}_1} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1}$, where $\hat{S}_{ii} = \frac{1}{T_i} \sum_t I_{i,t} \hat{\varepsilon}_{i,t}^2 x_t x_t'$, $\hat{\varepsilon}_{i,t} = R_{i,t} - \hat{\beta}_i' x_t$ and $c_{\hat{\nu}_1} = (1, -\hat{\nu}_1')'$. To estimate c_ν, we use the OLS estimator $\hat{\nu}_1 = \left(\sum_i \mathbf{1}_i^\chi \hat{b}_i \hat{b}_i' \right)^{-1} \sum_i \mathbf{1}_i^\chi \hat{b}_i \hat{a}_i$, i.e., a first-step estimator with unit weights. The WLS estimator is:

$$\hat{\nu} = \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{a}_i, \tag{5}$$

where $\hat{Q}_b = \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{b}_i'$ and $\hat{w}_i = \mathbf{1}_i^\chi \hat{v}_i^{-1}$. Weighting accounts for the statistical precision of the first-pass estimates. Under conditional homoskedasticity $\sigma_{ii,t} = \sigma_{ii}$ and a balanced panel $\tau_{i,T} = 1$, we have $v_i = c_\nu' Q_x^{-1} c_\nu \sigma_{ii}$. There, v_i is directly proportional to σ_{ii}, and we can simply pick the weights as $\hat{w}_i = \hat{\sigma}_{ii}^{-1}$, where $\hat{\sigma}_{ii} = \frac{1}{T} \sum_t \hat{\varepsilon}_{i,t}^2$ (Shanken (1992)). The final estimator of the risk premia is

$$\hat{\lambda} = \hat{\nu} + \frac{1}{T} \sum_t f_t. \tag{6}$$

Starting from the asset pricing restriction (4), another estimator of λ is $\bar{\lambda} = \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \bar{R}_i$, where $\bar{R}_i = \frac{1}{T_i} \sum_t I_{i,t} R_{i,t}$. This estimator is numerically equivalent to $\hat{\lambda}$ in the balanced case, where $I_{i,t} = 1$ for all i and t. In the general unbalanced case, it is equal to $\bar{\lambda} = \hat{\nu} + \hat{Q}_b^{-1} \frac{1}{n} \sum_i \hat{w}_i \hat{b}_i \hat{b}_i' \bar{f}_i$, where $\bar{f}_i = \frac{1}{T_i} \sum_t I_{i,t} f_t$. Estimator $\bar{\lambda}$ is often studied by the literature (see, e.g., Shanken (1992), Kandel and Stambaugh (1995), Jagannathan and Wang (1998)), and is also consistent. Estimating $E[f_t]$ with a simple average of the observed factor instead of a weighted average based on estimated betas simplifies the form of the asymptotic distribution in the unbalanced case (see below and Section 2.4). This explains our preference for $\hat{\lambda}$ over $\bar{\lambda}$.

We derive the asymptotic properties under assumptions on the conditional distribution of the error terms.

Assumption A.1 *There exists a positive constant M such that for all n:*
a) $E\left[\varepsilon_{i,t} | \{\varepsilon_{j,\underline{t-1}}, \gamma_j, j = 1, ..., n\}, x_{\underline{t}}\right] = 0$, *with* $\varepsilon_{i,\underline{t-1}} = \{\varepsilon_{i,t-1}, \varepsilon_{i,t-2}, \cdots\}$ *and* $x_{\underline{t}} = \{x_t, x_{t-1}, \cdots\}$;
b) $\sigma_{ii,t} \leq M, \quad i = 1, ..., n;$ *c)* $E\left[\frac{1}{n} \sum_{i,j} E\left[|\sigma_{ij,t}|^2 | \gamma_i, \gamma_j \right]^{1/2}\right] \leq M$, *where* $\sigma_{ij,t} = E\left[\varepsilon_{i,t} \varepsilon_{j,t} | x_{\underline{t}}, \gamma_i, \gamma_j\right]$.

Assumption A.1 allows for a martingale difference sequence for the error terms (White (2001)) including potential conditional heteroskedasticity as well as weak cross-sectional dependence (Bai and Ng (2002)).

More general error structures are possible but complicate consistent estimation of the asymptotic variances of the estimators (see Section 2.4).

Proposition 2 summarizes consistency of estimators $\hat{\nu}$ and $\hat{\lambda}$ under the double asymptotics $n, T \to \infty$. For sequences x_n and y_n, we denote $x_n \asymp y_n$ when x_n/y_n is bounded and bounded away from zero from below as $n \to \infty$.

Proposition 2 *Under Assumptions APR.1-APR.5, SC.1-SC.2, A.1 and C.1a), C.2-C.5, we get a)* $\|\hat{\nu} - \nu\| = o_p(1)$ *and b)* $\left\|\hat{\lambda} - \lambda\right\| = o_p(1)$*, when* $n, T \to \infty$ *such that* $n \asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma} > 0$*.*

The conditions in Proposition 2 allow for n large w.r.t. T (short panel asymptotics) when $\bar{\gamma} > 1$. Shanken (1992) shows consistency of $\hat{\nu}$ and $\hat{\lambda}$ for a fixed n and $T \to \infty$. This consistency does not imply Proposition 2. Shanken (1992) (see also Litzenberger and Ramaswamy (1979)) further shows that we can estimate ν consistently in the second pass with a modified cross-sectional estimator for a fixed T and $n \to \infty$. Since $\lambda = \nu + E[f_t]$, consistent estimation of the risk premia themselves is impossible for a fixed T (see Shanken (1992) for the same point).

Proposition 3 below gives the large-sample distributions under the double asymptotics $n, T \to \infty$. Let us define $\tau_{ij,T} = T/T_{ij}$, where $T_{ij} = \sum_t I_{ij,t}$ and $I_{ij,t} = I_{i,t}I_{j,t}$ for $i, j = 1, ..., n$. Let us further define $\tau_{ij} = \operatorname*{plim}_{T\to\infty} \tau_{ij,T} = E[I_{ij,t}|\gamma_i, \gamma_j]^{-1}$, $S_{ij} = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_t \sigma_{ij,t} x_t x_t' = E[\varepsilon_{i,t}\varepsilon_{j,t} x_t x_t' |\gamma_i, \gamma_j]$ and $Q_b = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_i w_i b_i b_i' = E[w_i b_i b_i']$. The following assumption describes the CLTs underlying the proof of the distributional properties. These CLTs hold under weak serial and cross-sectional dependencies such as temporal mixing and block dependence (see Appendix 4).

Assumption A.2 *As* $n, T \to \infty$ *such that* $n \asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma} \in \Gamma_1 \subset \mathbb{R}^+$*, a)* $\frac{1}{\sqrt{n}}\sum_i w_i \tau_i (Y_{i,T} \otimes b_i) \Rightarrow N(0, S_b)$*, where* $Y_{i,T} = \frac{1}{\sqrt{T}}\sum_t I_{i,t} x_t \varepsilon_{i,t}$ *and* $S_b = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i \tau_j}{\tau_{ij}} S_{ij} \otimes b_i b_j'\right] = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i \tau_j}{\tau_{ij}} S_{ij} \otimes b_i b_j'$*; b)* $\frac{1}{\sqrt{T}}\sum_t (f_t - E[f_t]) \Rightarrow N(0, \Sigma_f)$*, where* $\Sigma_f = \lim_{T\to\infty} \frac{1}{T}\sum_{t,s} Cov(f_t, f_s)$*.*

Proposition 3 *Under Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.2, and C.1a), C.2-C.5, we get: a)* $\sqrt{nT}\left(\hat{\nu} - \nu - \frac{1}{T}\hat{B}_\nu\right) \Rightarrow N(0, \Sigma_\nu)$*, where* $\Sigma_\nu = Q_b^{-1} \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i \tau_j}{\tau_{ij}} (c_\nu' Q_x^{-1} S_{ij} Q_x^{-1} c_\nu) b_i b_j'\right] Q_b^{-1}$

and the bias term is $\hat{B}_{\nu} = \hat{Q}_b^{-1}\left(\frac{1}{n}\sum_i \hat{w}_i \tau_{i,T} E_2' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}}\right)$*, with* $E_2 = (0 : Id_K)'$ *and* $c_{\hat{\nu}} = (1, -\hat{\nu}')'$*;*

b) $\sqrt{T}\left(\hat{\lambda} - \lambda\right) \Rightarrow N(0, \Sigma_f)$*, when* $n, T \to \infty$ *such that* $n \asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma} \in \Gamma_1 \cap (0, 3)$ *.*

The asymptotic variance matrix in Proposition 3 can be rewritten as:

$$\Sigma_{\nu} = \operatorname*{plim}_{n\to\infty}\left(\frac{1}{n}B_n' W_n B_n\right)^{-1} \frac{1}{n} B_n' W_n V_n W_n B_n \left(\frac{1}{n}B_n' W_n B_n\right)^{-1}$$

where $B_n = (b_1, ..., b_n)'$, $W_n = diag(w_1, ..., w_n)$ and $V_n = [v_{ij}]_{i,j=1,...,n}$ with $v_{ij} = \frac{\tau_i \tau_j}{\tau_{ij}} c_{\nu}' Q_x^{-1} S_{ij} Q_x^{-1} c_{\nu}$, which gives $v_{ii} = v_i$. In the homoskedastic and balanced case, we have $c_{\nu}' Q_x^{-1} c_{\nu} = 1 + \lambda' V[f_t]^{-1}\lambda$ and $V_n = (1 + \lambda' V[f_t]^{-1}\lambda)\Sigma_{\varepsilon,n}$, where $\Sigma_{\varepsilon,n} = [\sigma_{ij}]_{i,j=1,...,n}$. Then, the asymptotic variance of $\hat{\nu}$ reduces to $\operatorname*{plim}_{n\to\infty}(1 + \lambda' V[f_t]^{-1}\lambda)\left(\frac{1}{n}B_n' W_n B_n\right)^{-1} \frac{1}{n} B_n' W_n \Sigma_{\varepsilon,n} W_n B_n \left(\frac{1}{n}B_n' W_n B_n\right)^{-1}$. In particular, in the CAPM we have $K = 1$ and $\nu = 0$, which implies that $\sqrt{\frac{\lambda^2}{V[f_t]}}$ is equal to the slope of the Capital Market Line $\sqrt{\frac{E[f_t]^2}{V[f_t]}}$, i.e., the Sharpe Ratio of the market portfolio.

Proposition 3 shows that the estimator $\hat{\nu}$ has a fast convergence rate $\sqrt{nT}$ and features an asymptotic bias term. Both $\hat{a}_i$ and $\hat{b}_i$ in the definition of $\hat{\nu}$ contain an estimation error; for $\hat{b}_i$, this is the well-known Error-In-Variable (EIV) problem. The EIV problem does not impede consistency since we let T grow to infinity. However, it induces the bias term $\hat{B}_{\nu}/T$ which centers the asymptotic distribution of $\hat{\nu}$. We have $\Gamma_1 = \mathbb{R}^+$ in Assumption A.2, when $(\varepsilon_{i,t})$ and (x_t) are i.i.d. across time and errors $(\varepsilon_{i,t})$ feature a cross-sectional block dependence structure (see Appendix 4). Then, the upper bound on the relative expansion rates of n and T is $n = o(T^3)$. The control of first-pass estimation errors uniformly across assets requires that the cross-section dimension n should not be too large w.r.t. the time series dimension T.

If we knew the true factor mean, for example $E[f_t] = 0$, and did not need to estimate it, the estimator $\hat{\nu} + E[f_t]$ of the risk premia would have the same fast rate $\sqrt{nT}$ as the estimator of ν, and would inherit its asymptotic distribution. Since we do not know the true factor mean, the asymptotic distribution of $\hat{\lambda}$ is driven only by the variability of the factor since the convergence rate $\sqrt{T}$ of the sample average $\frac{1}{T}\sum_t f_t$ dominates the convergence rate $\sqrt{nT}$ of $\hat{\nu}$. This result is an oracle property for $\hat{\lambda}$, namely that its asymptotic distribution is the same irrespective of the knowledge of ν. This property is in sharp difference with the single asymptotics with a fixed n and $T \to \infty$. In the balanced case and with homoskedastic errors, Theorem 1 of Shanken (1992) shows that the rate of convergence of $\hat{\lambda}$ is $\sqrt{T}$ and that its asymptotic variance is $\Sigma_{\lambda,n} = \Sigma_f + (1 + \lambda' V[f_t]^{-1}\lambda) \left(\frac{1}{n}B_n' W_n B_n\right)^{-1} \frac{1}{n^2} B_n' W_n \Sigma_{\varepsilon,n} W_n B_n \left(\frac{1}{n}B_n' W_n B_n\right)^{-1}$, for fixed n and $T \to \infty$. The two components in $\Sigma_{\lambda,n}$ come from estimation of $E[f_t]$ and ν, respectively. In the heteroskedastic setting with fixed n, a slight extension of Theorem 1 in Jagannathan and Wang (1998), or Theorem 3.2 in Jagannathan, Skoulakis, and Wang (2009), to the unbalanced case yields $\Sigma_{\lambda,n} = \Sigma_f + \left(\frac{1}{n}B_n' W_n B_n\right)^{-1} \frac{1}{n^2} B_n' W_n V_n W_n B_n \left(\frac{1}{n}B_n' W_n B_n\right)^{-1}$. Letting $n \to \infty$ gives Σ_f under weak cross-sectional dependence. Thus, exploiting the full cross-section of assets improves efficiency asymptotically, and the positive definite matrix $\Sigma_{\lambda,n} - \Sigma_f$ corresponds to the efficiency gain. Using a large number of assets instead of a small number of portfolios does help to eliminate the EIV contribution.

Proposition 3 suggests exploiting the analytical bias correction $\hat{B}_\nu / T$ and using $\hat{\nu}_B = \hat{\nu} - \frac{1}{T}\hat{B}_\nu$ instead of $\hat{\nu}$. Furthermore, $\hat{\lambda}_B = \hat{\nu}_B + \frac{1}{T}\sum_t f_t$ delivers a bias-free estimator of λ at order $1/T$, which shares the same root-T asymptotic distribution as $\hat{\lambda}$.

Finally, we can relate the results of Proposition 3 to bias-corrected estimation accounting for the well-known incidental parameter problem of the panel literature (Neyman and Scott (1948), see Lancaster (2000) for a review). Model (1) under restriction (3) can be written as $R_{i,t} = b_i'(f_t + \nu) + \varepsilon_{i,t}$. In the likelihood setting of Hahn and Newey (2004) (see also Hahn and Kuersteiner (2002)), the b_i correspond to the individual effects and ν to the common parameter of interest. Available results tell us: (i) the estimator of ν is inconsistent if n goes to infinity while T is held fixed; (ii) the estimator of ν is asymptotically biased even if T grows at the same rate as n; (iii) an analytical bias correction may yield an estimator of ν that is root-(nT) asymptotically normal and centered at the truth if T grows faster than $n^{1/3}$. The two-pass estimators $\hat{\nu}$ and $\hat{\nu}_B$ exhibits the properties (i)-(iii) as expected by analogy with unbiased estimation in large panels. This clear link with the incidental parameter literature highlights another advantage of working with ν in the second pass regression.

2.4 Confidence intervals

We can use Proposition 3 to build confidence intervals by means of consistent estimation of the asymptotic variances. We can check with these intervals whether the risk of a given factor $f_{k,t}$ is not remunerated, i.e.,

$\lambda_k = 0$, or the restriction $\nu_k = 0$ holds when the factor is traded. We estimate Σ_f by a standard HAC estimator $\hat{\Sigma}_f$ such as in Newey and West (1994) or Andrews and Monahan (1992). Hence, the construction of confidence intervals with valid asymptotic coverage for components of $\hat{\lambda}$ is straightforward. On the contrary, getting a HAC estimator for $\bar{\Sigma}_f$ appearing in the asymptotic distribution of $\bar{\lambda}$ is not obvious in the unbalanced case.

The construction of confidence intervals for the components of $\hat{\nu}$ is more difficult. Indeed, Σ_ν involves a limiting double sum over S_{ij} scaled by n and not n^2. A naive approach consists in replacing S_{ij} by any consistent estimator such as $\hat{S}_{ij} = \frac{1}{T_{ij}} \sum_t I_{ij,t} \hat{\varepsilon}_{i,t} \hat{\varepsilon}_{j,t} x_t x_t'$, but this does not work here. To handle this, we rely on recent proposals in the statistical literature on consistent estimation of large-dimensional sparse covariance matrices by thresholding (Bickel and Levina (2008), El Karoui (2008)). Fan, Liao, and Mincheva (2011) have recently focused on the estimation of $E[\epsilon_t' \epsilon_t]$ in large balanced panel with nonrandom coefficients.

The idea is to assume sparse contributions of the S_{ij}'s to the double sum. Then we only have to account for sufficiently large contributions in the estimation, i.e., contributions larger than a threshold vanishing asymptotically. Thresholding permits an estimation invariant to asset permutations; this choice of estimator is motivated by the absence of any natural cross-sectional ordering among the matrices S_{ij}. In the following assumption we use the notion of sparsity suggested by Bickel and Levina (2008) adapted to our framework with random coefficients.

Assumption A.3 *There exist constants* $q, \delta \in [0, 1)$ *such that* $\max_i \sum_j \|S_{ij}\|^q = O_p\left(n^\delta\right)$.

Assumption A.3 tells us that most cross-asset contributions $\|S_{ij}\|$ can be neglected. As sparsity increases, we can choose coefficients q and δ closer to zero. Assumption A.3 does not impose sparsity of the covariance matrix of the returns themselves. Assumption A.1 c) is also a sparsity condition, which ensures that the limit matrix Σ_ν is well-defined when combined with Assumption C.3. Both sparsity assumptions are satisfied under weak cross-sectional dependence between the error terms, for instance, under a block dependence structure (see Appendix 4).

As in Bickel and Levina (2008), let us introduce the thresholded estimator $\tilde{S}_{ij} = \hat{S}_{ij} \mathbf{1} \left\{ \left\| \hat{S}_{ij} \right\| \geq \kappa \right\}$ of S_{ij}, which we refer to as $\hat{S}_{ij}$ thresholded at $\kappa = \kappa_{n,T}$. We can derive an asymptotically valid confidence interval for the components of $\hat{\nu}$ from the next proposition giving a feasible asymptotic normality result.

Proposition 4 *Under Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.3, C.1-C.5, we have* $\tilde{\Sigma}_{\nu}^{-1/2}\sqrt{nT}\left(\hat{\nu}-\frac{1}{T}\hat{B}_{\nu}-\nu\right)\Rightarrow N\left(0,Id_K\right)$ *where* $\tilde{\Sigma}_{\nu}=\hat{Q}_b^{-1}\left[\frac{1}{n}\sum_{i,j}\hat{w}_i\hat{w}_j\frac{\tau_{i,T}\tau_{j,T}}{\tau_{ij,T}}(c_{\hat{\nu}}'\hat{Q}_x^{-1}\tilde{S}_{ij}\hat{Q}_x^{-1}c_{\hat{\nu}})\hat{b}_i\hat{b}_j'\right]\hat{Q}_b^{-1}$, *when* $n,T\to\infty$ *such that* $n\asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma}\in\Gamma_1\cap\left(0,\min\left\{1+\eta,\eta\frac{1-q}{2\delta}\right\}\right)$, *and* $\kappa=M\sqrt{\frac{\log n}{T^{\eta}}}$ *for a constant* M *and* $\eta\in(0,1]$ *as in Assumption C.1.*

Constant $\eta\in(0,1]$ is defined in Assumption C.1 and is related to the time series dependence of processes $(\varepsilon_{i,t})$ and (x_t). We have $\eta=1$, when $(\varepsilon_{i,t})$ and (x_t) are serially i.i.d. as in Appendix 4 and Bickel and Levina (2008). The matrix made of thresholded blocks $\tilde{S}_{ij}$ is not guaranteed to be semi definite positive (sdp). However we expect that the double summation on i and j makes $\tilde{\Sigma}_{\nu}$ sdp in empirical applications. In case it is not, El Karoui (2008) discusses a few solutions based on shrinkage.

2.5 Tests of asset pricing restrictions

The null hypothesis underlying the asset pricing restriction (3) is

$$\mathcal{H}_0:\ \text{there exists } \nu\in\mathbb{R}^K \text{ such that } a(\gamma)=b(\gamma)'\nu,\quad \text{for almost all } \gamma\in[0,1].$$

Under $\mathcal{H}_0$, we have $E_G\left[(a_i-b_i'\nu)^2\right]=0$. Since ν is estimated via the WLS cross-sectional regression of the estimates $\hat{a}_i$ on the estimates $\hat{b}_i$, we suggest a test based on the weighted sum of squared residuals SSR of the cross-sectional regression. The weigthed SSR is $\hat{Q}_e=\frac{1}{n}\sum_i\hat{w}_i\hat{e}_i^2$, with $\hat{e}_i=c_{\hat{\nu}}'\hat{\beta}_i$, which is an empirical counterpart of $E_G\left[w_i\left(a_i-b_i'\nu\right)^2\right]$.

Let us define $S_{ii,T}=\frac{1}{T}\sum_t I_{i,t}\sigma_{ii,t}x_tx_t'$, and introduce the commutation matrix $W_{m,n}$ of order $mn\times mn$ such that $W_{m,n}vec\left[A\right]=vec\left[A'\right]$ for any matrix $A\in\mathbb{R}^{m\times n}$, where the vector operator $vec\left[\cdot\right]$ stacks the elements of an $m\times n$ matrix as a $mn\times 1$ vector. If $m=n$, we write W_n instead $W_{n,n}$. For two $(K+1)\times(K+1)$ matrices A and B, equality $W_{(K+1)}\left(A\otimes B\right)=\left(B\otimes A\right)W_{(K+1)}$ also holds (see Chapter 3 of Magnus and Neudecker (2007) for other properties).

Assumption A.4 *For* $n, T \to \infty$ *such that* $n \asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma} \in \Gamma_2 \subset \Gamma_1$*, we have* $\frac{1}{\sqrt{n}} \sum_i w_i \tau_i^2 \left(Y_{i,T} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]\right) \Rightarrow N\left(0, \Omega\right)$, *where the asymptotic variance matrix is:*

$$\begin{aligned} \Omega &= \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2}\left[S_{ij} \otimes S_{ij} + \left(S_{ij} \otimes S_{ij}\right) W_{(K+1)}\right]\right] \\ &= \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i,j} w_i w_j \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2}\left[S_{ij} \otimes S_{ij} + \left(S_{ij} \otimes S_{ij}\right) W_{(K+1)}\right]. \end{aligned}$$

Assumption A.4 is a high-level CLT condition. This assumption can be proved under primitive conditions on the time series and cross-sectional dependence. For instance, we prove in Appendix 4 that Assumption A.4 holds under a cross-sectional block dependence structure for the errors. Intuitively, the expression of the variance-covariance matrix Ω is related to the result that, for random $(K+1) \times 1$ vectors Y_1 and Y_2 which are jointly normal with covariance matrix S, we have $Cov\left(Y_1 \otimes Y_1, Y_2 \otimes Y_2\right) = S \otimes S + \left(S \otimes S\right) W_{(K+1)}$.

Let us now introduce the following statistic $\hat{\xi}_{nT} = T\sqrt{n}\left(\hat{Q}_e - \frac{1}{T}\hat{B}_\xi\right)$, where the recentering term simplifies to $\hat{B}_\xi = 1$ thanks to the weighting scheme. Under the null hypothesis $\mathcal{H}_0$, we prove that $\hat{\xi}_{nT} = \left(vec\left[\hat{Q}_x^{-1} c_{\hat{\nu}} c_{\hat{\nu}}' \hat{Q}_x^{-1}\right]\right)' \frac{1}{\sqrt{n}} \sum_i w_i \tau_i^2 \left(Y_{iT} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]\right) + o_p(1)$, which implies $\hat{\xi}_{nT} \Rightarrow N\left(0, \Sigma_\xi\right)$, where $\Sigma_\xi = 2 \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} w_i w_j v_{ij}^2\right] = 2 \operatorname*{plim}_{n\to\infty} \frac{1}{n} \sum_{i,j} w_i w_j v_{ij}^2$ as $n, T \to \infty$ (see Appendix A.2.5). Then a feasible testing procedure exploits the consistent estimator $\tilde{\Sigma}_\xi = 2\frac{1}{n}\sum_{i,j} \hat{w}_i \hat{w}_j \tilde{v}_{ij}^2$ of the asymptotic variance Σ_ξ, where $\tilde{v}_{ij} = \frac{\tau_{i,T}\tau_{j,T}}{\tau_{ij,T}} c_{\hat{\nu}}' \hat{Q}_x^{-1} \tilde{S}_{ij} \hat{Q}_x^{-1} c_{\hat{\nu}}$.

Proposition 5 *Under* $\mathcal{H}_0$*, and Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.4 and C.1-C.5, we have* $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT} \Rightarrow N\left(0, 1\right)$*, as* $n, T \to \infty$ *such that* $n \asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma} \in \Gamma_2 \cap \left(0, \min\left\{2\eta, \eta\frac{1-q}{2\delta}\right\}\right)$.

In the homoskedastic case, the asymptotic variance of $\hat{\xi}_{nT}$ reduces to $\Sigma_\xi = 2 \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}^2} \frac{\sigma_{ij}^2}{\sigma_{ii}\sigma_{jj}}$. For fixed n, we can rely on the test statistic $T\hat{Q}_e$, which is asymptotically distributed as $\frac{1}{n}\sum_j eig_j \chi_j^2$ for $j = 1, \ldots, (n-K)$, where the χ_j^2 are i.i.d. chi-square variables with 1 degree of freedom, and the coefficients eig_j are the non-zero eigenvalues of matrix $V_n^{1/2}(W_n - W_n B_n (B_n' W_n B_n)^{-1} B_n' W_n) V_n^{1/2}$ (see

Kan et al. (2009)). By letting n grow, the sum of chi-square variables converges to a Gaussian variable after recentering and rescaling, which yields heuristically the result of Proposition 5.

The alternative hypothesis is

$$\mathcal{H}_1: \quad \inf_{\nu \in \mathbb{R}^K} E_G \left[\left(a_i - b_i' \nu \right)^2 \right] > 0.$$

Let us define the pseudo-true value $\nu_\infty = \arg \inf_{\nu \in \mathbb{R}^K} Q_\infty^w(\nu)$, where $Q_\infty^w(\nu) = E_G \left[w_i \left(a_i - b_i' \nu \right)^2 \right]$ (White (1982), Gourieroux, Monfort and Trognon (1984)) and population errors $e_i = a_i - b_i' \nu_\infty = c_{\nu_\infty}' \beta_i$, $i = 1, ..., n$, for all n. In the next proposition, we prove consistency of the test, namely that the statistic $\hat{\xi}_{nT}$ diverges to $+\infty$ under the alternative hypothesis $\mathcal{H}_1$ for large n and T. We also give the asymptotic distribution of estimators $\hat{\nu}$ and $\hat{\lambda}$ under $\mathcal{H}_1$.

Proposition 6 *Under $\mathcal{H}_1$ and Assumptions APR.1-APR.5, SC.1-SC.2, A.1-A.4 and C.1-C.5, we have $\hat{\xi}_{nT} \xrightarrow{p} +\infty$, and $\sqrt{n} \left(\hat{\nu} - \nu_\infty \right) \Rightarrow N\left(0, \Sigma_{\nu_\infty}\right)$, where $\Sigma_{\nu_\infty} = Q_b^{-1} E_G[w_i^2 e_i^2 b_i b_i'] Q_b^{-1}$ and $\sqrt{T} \left(\hat{\lambda} - \lambda_\infty \right) \Rightarrow N\left(0, \Sigma_f\right)$, and $\lambda_\infty = \nu_\infty + E\left[f_t\right]$, as $n, T \to \infty$ such that $n \asymp T^{\bar{\gamma}}$ for $\bar{\gamma} \in \Gamma_2 \cap \left(1, \min\left\{2\eta, \eta \dfrac{1-q}{2\delta}\right\}\right)$.*

Under the alternative hypothesis $\mathcal{H}_1$, the rate of convergence of $\hat{\nu}$ is slower than under $\mathcal{H}_0$, while the rate of convergence of $\hat{\lambda}$ remains the same. The asymptotic distribution of $\hat{\nu}$ is the same as the one got from a cross-sectional regression of a_i on b_i. Pre-estimation of b_i has no impact on the asymptotic distribution of $\hat{\nu}$ since the bias induced by the EIV problem is of the order $O(1/T)$, and $\sqrt{n}/T = o(1)$. The lower bound 1 on rate $\bar{\gamma}$ in Proposition 6 ensures that cross-sectional estimation of ν has asymptotically no impact on the estimation of λ.

To study the local asymptotic power, we can adopt the following local alternative: $\mathcal{H}_{1,nT}: \quad \inf_{\nu \in \mathbb{R}^K} Q_\infty^w(\nu) = \dfrac{\psi}{\sqrt{n}T} > 0$, for a constant $\psi > 0$. Then we can show (see the supplementary materials) that $\hat{\xi}_{nT} \Rightarrow N(\psi, \Sigma_\xi)$, and the test is locally asymptotically powerful. Pesaran and Yamagata (2008) consider a similar local analysis for a test of slope homogeneity in large panels.

Finally, we can derive a test for the null hypothesis when the factors come from tradable assets, i.e., are portfolio excess returns:

$$\mathcal{H}_0: a(\gamma) = 0 \text{ for almost all } \gamma \in [0, 1] \quad \Leftrightarrow \quad E_G[a_i^2] = 0,$$

against the alternative hypothesis

$$\mathcal{H}_1 : E_G\left[a_i^2\right] > 0.$$

We only have to substitute $\hat{a}_i$ for $\hat{e}_i$, and $E_1 = (1, 0')'$ for $c_{\hat{\nu}}$ in Proposition 5.

3 Conditional factor model

In this section we extend the setting of Section 2 to conditional specifications in order to model possibly time-varying risk premia (see Connor and Korajczyk (1989) for an intertemporal competitive equilibrium version of the APT yielding time-varying risk premia and Ludvigson (2011) for a discussion within scaled consumption-based models). We do not follow rolling short-window regression approaches to account for time-variation (Fama and French (1997), Lewellen and Nagel (2006)) since we favor a structural econometric framework to conduct formal inference in large cross-sectional equity datasets. A five-year window of monthly data yields a very short time-series panel for which asymptotics with fixed (small) T and large n are better suited, but keeping T fixed impedes consistent estimation of the risk premia as already mentioned in the previous section.

3.1 Excess return generation and asset pricing restrictions

The following assumptions are the analogues of Assumptions APR.1 and APR.2, and Proposition 7 is the analogue of Proposition 1.

Assumption APR.6 *The excess returns $R_t(\gamma)$ of asset $\gamma \in [0,1]$ at date $t = 1, 2, ...$ satisfy the conditional linear factor model:*

$$R_t(\gamma) = a_t(\gamma) + b_t(\gamma)' f_t + \varepsilon_t(\gamma), \tag{7}$$

where $a_t(\gamma, \omega) = a[\gamma, S^{t-1}(\omega)]$ and $b_t(\gamma, \omega) = b[\gamma, S^{t-1}(\omega)]$, for any $\omega \in \Omega$ and $\gamma \in [0,1]$, and random variable $a(\gamma)$ and random vector $b(\gamma)$, for $\gamma \in [0,1]$, are $\mathcal{F}_0$-measurable.

The intercept $a_t(\gamma)$ and factor sensitivity $b_t(\gamma)$ of asset $\gamma \in [0,1]$ at time t are $\mathcal{F}_{t-1}$-measurable.

Assumption APR.7 *The matrix* $\int b_t(\gamma)b_t(\gamma)'d\gamma$ *is positive definite,* P*-a.s., for any date* $t = 1, 2, \ldots$.

Proposition 7 *Under Assumptions APR.3-APR.7, for any date* $t = 1, 2, \ldots$ *there exists a unique random vector* $\nu_t \in \mathbb{R}^K$ *such that* ν_t *is* $\mathcal{F}_{t-1}$*-measurable and:*

$$a_t(\gamma) = b_t(\gamma)'\nu_t, \tag{8}$$

P*-a.s. and for almost all* $\gamma \in [0, 1]$.

The asset pricing restriction in Proposition 7 can be rewritten as

$$E\left[R_t(\gamma)|\mathcal{F}_{t-1}\right] = b_t(\gamma)'\lambda_t, \tag{9}$$

for almost all $\gamma \in [0, 1]$, where $\lambda_t = \nu_t + E\left[f_t|\mathcal{F}_{t-1}\right]$ is the vector of the conditional risk premia.

To have a workable version of equations (7) and (9), we further specify the conditioning information and how coefficients depend on it. The conditioning information is such that $\mathcal{F}_t = \{S^{-t}(A), A \in \mathcal{F}_0\}$, $t = 1, 2, \ldots$, and instruments $Z \in \mathbb{R}^p$ and $Z(\gamma) \in \mathbb{R}^q$, for $\gamma \in [0, 1]$, are $\mathcal{F}_0$-measurable. Then, the information $\mathcal{F}_{t-1}$ contain $Z_{\underline{t-1}}$ and $Z_{\underline{t-1}}(\gamma)$, for $\gamma \in [0, 1]$, where we define $Z_t(\omega) = Z[S^t(\omega)]$ and $Z_t(\gamma, \omega) = Z[\gamma, S^t(\omega)]$. The lagged instruments Z_{t-1} are common to all stocks. They may include the constant and past observations of the factors and some additional variables such as macroeconomic variables. The lagged instruments $Z_{t-1}(\gamma)$ are specific to stock γ. They may include past observations of firm characteristics and stock returns. To end up with a linear regression model we specify that the vector of factor sensitivities $b_t(\gamma)$ is a linear function of lagged instruments Z_{t-1} (Shanken (1990), Ferson and Harvey (1991)) and $Z_{t-1}(\gamma)$ (Avramov and Chordia (2006)): $b_t(\gamma) = B(\gamma)Z_{t-1} + C(\gamma)Z_{t-1}(\gamma)$, where $B(\gamma) \in \mathbb{R}^{K\times p}$ and $C(\gamma) \in \mathbb{R}^{K\times q}$, for any $\gamma \in [0, 1]$ and $t = 1, 2, \ldots$. We can account for nonlinearities by including powers of some explanatory variables among the lagged instruments. We also specify that the vector of risk premia is a linear function of lagged instruments Z_{t-1} (Cochrane (1996), Jagannathan and Wang (1996)): $\lambda_t = \Lambda Z_{t-1}$, where $\Lambda \in \mathbb{R}^{K\times p}$, for any t. Furthermore, we assume that the conditional expectation of Z_t given the information $\mathcal{F}_{t-1}$ depends on Z_{t-1} only and is linear, as, for instance, in an exogeneous Vector Autoregressive (VAR) model of order 1. Since f_t is a subvector of Z_t, then $E\left[f_t|\mathcal{F}_{t-1}\right] = FZ_{t-1}$, where $F \in \mathbb{R}^{K\times p}$, for any

t. Under these functional specifications the asset pricing restriction (9) implies that the intercept $a_t(\gamma)$ is a quadratic form in lagged instruments Z_{t-1} and $Z_{t-1}(\gamma)$, namely:

$$a_t(\gamma) = Z'_{t-1}B(\gamma)' (\Lambda - F) Z_{t-1} + Z_{t-1}(\gamma)'C(\gamma)' (\Lambda - F) Z_{t-1}. \qquad (10)$$

This shows that assuming a priori linearity of $a_t(\gamma)$ in the lagged instruments Z_{t-1} and $Z_{t-1}(\gamma)$ is in general not compatible with linearity of $b_t(\gamma)$ and $E[f_t|Z_{t-1}]$.

The sampling scheme is the same as in Section 2.2, and we use the same type of notation, for example $b_{i,t} = b_t(\gamma_i)$, $B_i = B(\gamma_i)$, $C_i = C(\gamma_i)$ and $Z_{i,t-1} = Z_{t-1}(\gamma_i)$. Then, the conditional factor model (7) with asset pricing restriction (10) written for the sample observations becomes

$$R_{i,t} = Z'_{t-1}B'_i (\Lambda - F) Z_{t-1} + Z'_{i,t-1}C'_i (\Lambda - F) Z_{t-1} + Z'_{t-1}B'_i f_t + Z'_{i,t-1}C'_i f_t + \varepsilon_{i,t}, \qquad (11)$$

which is nonlinear in the parameters Λ, F, B_i, and C_i. In order to implement the two-pass methodology in a conditional context it is useful to rewrite model (11) as a model that is linear in transformed parameters and new regressors. The regressors include $x_{2,i,t} = \left(f'_t \otimes Z'_{t-1}, f'_t \otimes Z'_{i,t-1}\right)' \in \mathbb{R}^{d_2}$ with $d_2 = K(p+q)$. The first components with common instruments take the interpretation of scaled factors, while the second components do not since they depend on i. The regressors also include the predetermined variables $x_{1,i,t} = \left(vech\,[X_t]', vec\,[X_{i,t}]'\right)' \in \mathbb{R}^{d_1}$ with $d_1 = p(p+1)/2 + pq$, where the symmetric matrix $X_t = [X_{t,k,l}] \in \mathbb{R}^{p\times p}$ is such that $X_{t,k,l} = Z^2_{t-1,k}$, if $k = l$, and $X_{t,k,l} = 2Z_{t-1,k}Z_{t-1,l}$, otherwise, $k, l = 1, \ldots, p$, and the matrix $X_{i,t} = Z_{t-1}Z'_{i,t-1} \in \mathbb{R}^{p\times q}$. The vector-half operator $vech\,[\cdot]$ stacks the lower elements of a $p \times p$ matrix as a $p\,(p+1)\,/2 \times 1$ vector (see Chapter 2 in Magnus and Neudecker (2007) for properties of this matrix tool). To parallel the analysis of the unconditional case, we can express model (11) as in (2) through appropriate redefinitions of the regressors and loadings (see Appendix 3):

$$R_{i,t} = \beta'_i x_{i,t} + \varepsilon_{i,t}, \qquad (12)$$

where $x_{i,t} = \left(x'_{1,i,t}, x'_{2,i,t}\right)'$ has dimension $d = d_1 + d_2$, and $\beta_i = \left(\beta'_{1,i}, \beta'_{2,i}\right)'$ is such that

$$\beta_{1,i} = \Psi\beta_{2,i}, \qquad \beta_{2,i} = \left(vec\left[B'_i\right]', vec\left[C'_i\right]'\right)', \qquad (13)$$

$$\Psi = \begin{pmatrix} \frac{1}{2}D^+_p\left[(\Lambda - F)' \otimes I_p + I_p \otimes (\Lambda - F)'W_{p,K}\right] & 0 \\ 0 & (\Lambda - F)' \otimes I_q \end{pmatrix}.$$

The matrix D_p^+ is the $p(p+1)/2 \times p^2$ Moore-Penrose inverse of the duplication matrix D_p, such that $vech\,[A] = D_p^+ vec\,[A]$ for any $A \in \mathbb{R}^{p\times p}$ (see Chapter 3 in Magnus and Neudecker (2007)). When $Z_t = 1$ and $Z_{i,t} = 0$, we have $p = p(p+1)/2 = 1$ and $q = 0$, and model (12) reduces to model (2).

In (13), the $d_1 \times 1$ vector $\beta_{1,i}$ is a linear transformation of the $d_2 \times 1$ vector $\beta_{2,i}$. This clarifies that the asset pricing restriction (10) implies a constraint on the distribution of random vector β_i via its support. The coefficients of the linear transformation depend on matrix $\Lambda - F$. For the purpose of estimating the loading coefficients of the risk premia in matrix Λ, the parameter restrictions can be written as (see Appendix 3):

$$\beta_{1,i} = \beta_{3,i}\nu, \qquad \nu = vec\left[\Lambda' - F'\right], \qquad \beta_{3,i} = \left(\left[D_p^+\left(B_i' \otimes I_p\right)\right]', \left[W_{p,q}\left(C_i' \otimes I_p\right)\right]'\right)'. \tag{14}$$

Furthermore, we can relate the $d_1 \times Kp$ matrix $\beta_{3,i}$ to the vector $\beta_{2,i}$ (see Appendix 3):

$$vec\left[\beta_{3,i}'\right] = J_a \beta_{2,i}, \tag{15}$$

where the $d_1pK \times d_2$ block-diagonal matrix of constants J_a is given by $J_a = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix}$ with diagonal blocks $J_{11} = W_{p(p+1)/2,pK}\left(I_K \otimes \left[\left(I_p \otimes D_p^+\right)\left(W_p \otimes I_p\right)\left(I_p \otimes vec\left[I_p\right]\right)\right]\right)$ and $J_{22} = W_{pq,pK}\left(I_K \otimes \left[\left(I_p \otimes W_{p,q}\right)\left(W_{p,q} \otimes I_p\right)\left(I_q \otimes vec\left[I_p\right]\right)\right]\right)$. The link (15) is instrumental in deriving the asymptotic results. The parameters $\beta_{1,i}$ and $\beta_{2,i}$ correspond to the parameters a_i and b_i of the unconditional case, where the matrix J_a is equal to I_K. Equations (14) and (15) in the conditional setting are the counterparts of restriction (3) in the static setting.

3.2 Asymptotic properties of time-varying risk premium estimation

We consider a two-pass approach building on Equations (12) and (14).

First Pass: The first pass consists in computing time-series OLS estimators $\hat{\beta}_i = (\hat{\beta}_{1,i}', \hat{\beta}_{2,i}')' = \hat{Q}_{x,i}^{-1} \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} R_{i,t}$, for $i = 1, ..., n$, where $\hat{Q}_{x,i} = \frac{1}{T_i} \sum_t I_{i,t} x_{i,t} x_{i,t}'$. We use the same trimming device as in Section 2.

Second Pass: The second pass consists in computing a cross-sectional estimator of ν by regressing the $\hat{\beta}_{1,i}$ on the $\hat{\beta}_{3,i}$ keeping non-trimmed assets only. We use a WLS approach. The weights are estimates of $w_i = (diag\,[v_i])^{-1}$, where the v_i are the asymptotic variances of the standardized errors $\sqrt{T}\left(\hat{\beta}_{1,i} - \hat{\beta}_{3,i}\nu\right)$ in the cross-sectional regression for large T. We have $v_i = \tau_i C_\nu' Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1} C_\nu$, where $Q_{x,i} = E\left[x_{i,t}x_{i,t}'|\gamma_i\right]$,

$S_{ii} = \underset{T\to\infty}{\text{plim}} \frac{1}{T}\sum_t \sigma_{ii,t} x_{i,t} x_{i,t}' = E\left[\varepsilon_{i,t}^2 x_{i,t} x_{i,t}' | \gamma_i\right]$, $\sigma_{ii,t} = E\left[\varepsilon_{i,t}^2 | x_{i,\underline{t}}, \gamma_i\right]$, and $C_\nu = \left(E_1' - \left(I_{d_1} \otimes \nu'\right) J_a E_2'\right)'$, with $E_1 = (I_{d_1}, 0_{d_1 \times d_2})'$, $E_2 = (0_{d_2 \times d_1}, I_{d_2})'$. We use the estimates $\hat{v}_i = \tau_{i,T} C_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}_1}$, where $\hat{S}_{ii} = \frac{1}{T_i}\sum_t I_{i,t} \hat{\varepsilon}_{i,t}^2 x_{i,t} x_{i,t}'$, $\hat{\varepsilon}_{i,t} = R_{i,t} - \hat{\beta}_i' x_{i,t}$ and $C_{\hat{\nu}_1} = \left(E_1' - \left(I_{d_1} \otimes \hat{\nu}_1'\right) J_a E_2'\right)'$. To estimate C_ν, we use the OLS estimator $\hat{\nu}_1 = \left(\sum_i \mathbf{1}_i^\chi \hat{\beta}_{3,i}' \hat{\beta}_{3,i}\right)^{-1} \sum_i \mathbf{1}_i^\chi \hat{\beta}_{3,i}' \hat{\beta}_{1,i}$, i.e., a first-step estimator with unit weights. The WLS estimator is:

$$\hat{\nu} = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}_{3,i}' \hat{w}_i \hat{\beta}_{1,i}, \tag{16}$$

where $\hat{Q}_{\beta_3} = \frac{1}{n}\sum_i \hat{\beta}_{3,i}' \hat{w}_i \hat{\beta}_{3,i}$ and $\hat{w}_i = \mathbf{1}_i^\chi \left(diag\left[\hat{v}_i\right]\right)^{-1}$. The final estimator of the risk premia is $\hat{\lambda}_t = \hat{\Lambda} Z_{t-1}$ where we deduce $\hat{\Lambda}$ from the relationship $vec\left[\hat{\Lambda}'\right] = \hat{\nu} + vec\left[\hat{F}'\right]$ with the estimator $\hat{F}$ obtained by a SUR regression of factors f_t on lagged instruments Z_{t-1}: $\hat{F} = \sum_t f_t Z_{t-1}' \left(\sum_t Z_{t-1} Z_{t-1}'\right)^{-1}$.

The next assumption is similar to Assumption A.1.

Assumption B.1 *There exists a positive constant M such that for all n, T:*
a) $E\left[\varepsilon_{i,t} | \{\varepsilon_{j,\underline{t-1}}, Z_{j,\underline{t-1}}, j = 1,...,n\}, Z_{\underline{t}}\right] = 0$, *with* $Z_{\underline{t}} = \{Z_t, Z_{t-1}, \cdots\}$ *and* $Z_{j,\underline{t}} = \{Z_{j,t}, Z_{j,t-1}, \cdots\}$
b) $\sigma_{ii,t} \leq M,\ i = 1,...,n;$ *c)* $E\left[\frac{1}{n}\sum_{i,j} E\left[\left|\sigma_{ij,t}\right|^2 | \gamma_i, \gamma_j\right]^{1/2}\right] \leq M$, *where* $\sigma_{ij,t} = E\left[\varepsilon_{i,t}\varepsilon_{j,t} | x_{i,\underline{t}}, x_{j,\underline{t}}, \gamma_i, \gamma_j\right]$.

Proposition 8 summarizes consistency of estimators $\hat{\nu}$ and $\hat{\Lambda}$ under the double asymptotics $n, T \to \infty$. It extends Proposition 2 to the conditional case.

Proposition 8 *Under Assumptions APR.3-APR.7, SC.1-SC.2, B.1 and C.1a), C.2-C.6, we get a)* $\|\hat{\nu} - \nu\| = o_p(1)$, *b)* $\left\|\hat{\Lambda} - \Lambda\right\| = o_p(1)$, *when $n, T \to \infty$ such that $n \asymp T^{\bar{\gamma}}$ for $\bar{\gamma} > 0$.*

Part b) implies $\sup_t \left\|\hat{\lambda}_t - \lambda_t\right\| = o_p(1)$ under for instance a boundeness assumption on process Z_t.

Proposition 9 below gives the large-sample distributions under the double asymptotics $n, T \to \infty$. It extends Proposition 3 to the conditional case through adequate use of selection matrices. The following assumption is similar to Assumption A.2. We make use of $Q_{\beta_3} = E_G\left[\beta_{3,i}' w_i \beta_{3,i}\right]$, $Q_z = E\left[Z_t Z_t'\right]$, $S_{ij} = \underset{T\to\infty}{\text{plim}} \frac{1}{T}\sum_t \sigma_{ij,t} x_{i,t} x_{j,t}' = E[\varepsilon_{i,t}\varepsilon_{j,t} x_{i,t} x_{j,t}' | \gamma_i, \gamma_j]$ and $S_{Q,ij} = Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1}$, otherwise, we keep the same notations as in Section 2.

Assumption B.2 *As $n, T \to \infty$ such that $n \asymp T^{\bar{\gamma}}$ for $\bar{\gamma} \in \Gamma_1 \subset \mathbb{R}^+$, a)* $\frac{1}{\sqrt{n}} \sum_i \tau_i \left[(Q_{x,i}^{-1} Y_{i,T}) \otimes v_{3,i} \right] \Rightarrow N(0, S_{v_3})$*, with* $Y_{i,T} = \frac{1}{\sqrt{T}} \sum_t I_{i,t} x_{i,t} \varepsilon_{i,t}$*,* $v_{3,i} = vec[\beta_{3,i}' w_i]$ *and* $S_{v_3} = \lim_{n\to\infty} E \left[\frac{1}{n} \sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}} S_{Q,ij} \otimes v_{3,i} v_{3,j}' \right] = \text{plim}_{n\to\infty} \frac{1}{n} \sum_{i,j} \frac{\tau_i \tau_j}{\tau_{ij}} [S_{Q,ij} \otimes v_{3,i} v_{3,j}']$*; b)* $\frac{1}{\sqrt{T}} \sum_t u_t \otimes Z_{t-1} \Rightarrow N(0, \Sigma_u)$*, where* $\Sigma_u = E\left[u_t u_t' \otimes Z_{t-1} Z_{t-1}' \right]$ *and* $u_t = f_t - F Z_{t-1}$.

Proposition 9 *Under Assumptions APR.3-APR.7,SC.1-SC.2,B.1-B.2 and C.1a), C.2-C.6, we have a)* $\sqrt{nT} \left(\hat{\nu} - \nu - \frac{1}{T} \hat{B}_\nu \right) \Rightarrow N(0, \Sigma_\nu)$ *where* $\hat{B}_\nu = \hat{Q}_{\beta_3}^{-1} J_b \frac{1}{n} \sum_i \tau_{i,T} vec \left[E_2' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i \right]$ *and* $\Sigma_\nu = \left(vec\left[C_\nu' \right] \otimes Q_{\beta_3}^{-1} \right)' S_{v_3} \left(vec\left[C_\nu' \right] \otimes Q_{\beta_3}^{-1} \right)$*, with* $J_b = \left(vec\left[I_{d_1} \right]' \otimes I_{Kp} \right) \left(I_{d_1} \otimes J_a \right)$ *and* $C_{\hat{\nu}} = \left(E_1' - \left(I_{d_1} \otimes \hat{\nu}' \right) J_a E_2' \right)'$*; b)* $\sqrt{T} vec \left[\hat{\Lambda}' - \Lambda' \right] \Rightarrow N(0, \Sigma_\Lambda)$ *where* $\Sigma_\Lambda = \left(I_K \otimes Q_z^{-1} \right) \Sigma_u \left(I_K \otimes Q_z^{-1} \right)$*, when $n, T \to \infty$ such that $n \asymp T^\gamma$ for $\gamma \in \Gamma_1 \cap (0, 3)$.*

Since $\lambda_t = \Lambda Z_{t-1} = \left(Z_{t-1}' \otimes I_K \right) W_{p,K} vec\left[\Lambda' \right]$, part b) implies conditionally on Z_{t-1} that $\sqrt{T} \left(\hat{\lambda}_t - \lambda_t \right) \Rightarrow N\left(0, \left(Z_{t-1}' \otimes I_K \right) W_{p,K} \Sigma_\Lambda W_{K,p} \left(Z_{t-1} \otimes I_K \right) \right)$.

We can use Proposition 9 to build confidence intervals. It suffices to replace the unknown quantities Q_x, Q_z, Q_{β_3}, Σ_u and ν by their empirical counterparts. For matrix S_{v_3} we use the thresholded estimator $\tilde{S}_{ij}$ as in Section 2.4. Then we can extend Proposition 4 to the conditional case under Assumptions B.1-B.2, A.3, A.4 and C.1-C.6.

Since Equation (14) corresponds to the asset pricing restriction (3), the null hypothesis of correct specification of the conditional model is

$$\mathcal{H}_0: \text{ there exists } \nu \in \mathbb{R}^{pK} \text{ such that } \beta_1(\gamma) = \beta_3(\gamma)\nu, \text{ with } vec\left[\beta_3(\gamma)' \right] = J_a \beta_2(\gamma), \text{ for almost all } \gamma \in [0, 1].$$

Under $\mathcal{H}_0$, we have $E_G \left[(\beta_{1,i} - \beta_{3,i}\nu)' (\beta_{1,i} - \beta_{3,i}\nu) \right] = 0$. The alternative hypothesis is

$$\mathcal{H}_1: \quad \inf_{\nu \in \mathbb{R}^{dK}} E_G \left[(\beta_{1,i} - \beta_{3,i}\nu)' (\beta_{1,i} - \beta_{3,i}\nu) \right] > 0.$$

As in Section 2.5, we build the SSR $\hat{Q}_e = \frac{1}{n} \sum_i \hat{e}_i' \hat{w}_i \hat{e}_i$, with $\hat{e}_i = \hat{\beta}_{1,i} - \hat{\beta}_{3,i}\hat{\nu} = C_{\hat{\nu}}' \hat{\beta}_i$ and the statistic $\hat{\xi}_{nT} = T\sqrt{n} \left(\hat{Q}_e - \frac{1}{T} \hat{B}_\xi \right)$, where $\hat{B}_\xi = d_1$.

Assumption B.3 *For* $n, T \rightarrow \infty$ *such that* $n \asymp T^{\bar{\gamma}}$ *for* $\bar{\gamma} \in \Gamma_2 \subset \Gamma_1$*, we have* $\frac{1}{\sqrt{n}} \sum_i \tau_i^2 \left[\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1} \right) (Y_{i,T} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]) \right] \otimes vec[w_i] \Rightarrow N\left(0, \Omega\right)$, *where the asymptotic variance matrix is:*

$$
\begin{aligned}
\Omega &= \lim_{n\to\infty} E\left[\frac{1}{n} \sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} \left[S_{Q,ij} \otimes S_{Q,ij} + \left(S_{Q,ij} \otimes S_{Q,ij} \right) W_d \right] \otimes \left(vec[w_i] vec[w_j]' \right) \right] \\
&= \operatorname*{plim}_{n\to\infty} \frac{1}{n} \sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} \left[S_{Q,ij} \otimes S_{Q,ij} + \left(S_{Q,ij} \otimes S_{Q,ij} \right) W_d \right] \otimes \left(vec[w_i] vec[w_j]' \right).
\end{aligned}
$$

Proposition 10 *Under* $\mathcal{H}_0$ *and Assumptions APR.3-APR.7, SC.1-SC.2, B.1-B.2, A.3, A.4 and C.1-C.6, we have* $\tilde{\Sigma}_\xi^{-1/2} \hat{\xi}_{nT} \Rightarrow N\left(0,1\right)$, *where* $\tilde{\Sigma}_\xi = 2 \frac{1}{n} \sum_{i,j} \frac{\tau_{i,T}^2 \tau_{j,T}^2}{\tau_{ij,T}^2} tr \left[\hat{w}_i \left(C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \tilde{S}_{ij} \hat{Q}_{x,j}^{-1} C_{\hat{\nu}} \right) \hat{w}_j \left(C_{\hat{\nu}}' \hat{Q}_{x,j}^{-1} \tilde{S}_{ji} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \right) \right]$ *as* $n, T \to \infty$ *such that* $n \asymp T^\gamma$ *for* $\gamma \in \Gamma_2 \cap \left(0, \min\left\{ 2\eta, \eta \frac{1-q}{2\delta} \right\} \right)$.

Under $\mathcal{H}_1$, we have $\hat{\xi} \xrightarrow{p} +\infty$, as in Proposition 6.

As in Section 2.5, the null hypothesis when the factors are tradable assets becomes:

$$
\mathcal{H}_0: \quad \beta_1(\gamma) = 0 \text{ for almost all } \gamma \in [0,1],
$$

against the alternative hypothesis

$$
\mathcal{H}_1: \quad E_G\left[\beta_{1,i}' \beta_{1,i}\right] > 0.
$$

We only have to substitute $\hat{Q}_a = \frac{1}{n} \sum_i \hat{\beta}_{1,i}' \hat{w}_i \hat{\beta}_{1,i}$ for $\hat{Q}_e$, and $E_1 = (I_{d_1} : 0)'$ for $C_{\hat{\nu}}$. This gives an extension of Gibbons, Ross and Shanken (1989) to the conditional case and with double asymptotics. Implementing the original Gibbons, Ross and Shanken (1989) test, which uses a weighting matrix corresponding to an inverted estimated covariance matrix, becomes quickly problematic; each $\beta_{1,i}$ is of dimension $d_1 \times 1$, and the inverted matrix is of dimension $nd_1 \times nd_1$. We expect to compensate the potential loss of power induced by a diagonal weighting thanks to the large number nd_1 of restrictions. Our preliminary unreported Monte Carlo simulations show that the test exhibits good power properties for a couple of hundreds of assets.

4 Empirical results

4.1 Asset pricing model and data description

Our baseline asset pricing model is a four-factor model with $f_t = (r_{m,t}, r_{smb,t}, r_{hml,t}, r_{mom,t})'$ where $r_{m,t}$ is the month t excess return on CRSP NYSE/AMEX/Nasdaq value-weighted market portfolio over the risk free rate (proxied by the monthly 30-day T-bill beginning-of-month yield), and $r_{smb,t}$, $r_{hml,t}$ and $r_{mom,t}$ are the month t returns on zero-investment factor-mimicking portfolios for size, book-to-market, and momentum (see Fama and French (1993), Jegadeesh and Titman (1993), Carhart (1997)). To account for time-varying alphas, betas and risk premia, we use a specification based on two common variables and two firm-level variables. We take the instruments $Z_t = (1, Z_t^{*\prime})'$, where bivariate vector Z_t^* includes the term spread, proxied by the difference between yields on 10-year Treasury and three-month T-bill, and the default spread, proxied by the yield difference between Moody's Baa-rated and Aaa-rated corporate bonds. We take $Z_{i,t}$ as a bivariate vector made of the market capitalization and the book-to-market equity of firm i. We refer to Avramov and Chordia (2006) for convincing theoretical and empirical arguments in favor of the chosen conditional specification. The vector $x_{i,t}$ is of dimension $d = 32$. The firm characteristics are computed as in the appendix of Fama and French (2008) from Compustat. We use monthly stock returns data provided by CRSP and we exclude financial firms (Standard Industrial Classification Codes between 6000 and 6999) as in Fama and French (2008). The dataset after matching CRSP and Compustat contents comprises $n = 9,936$ stocks and covers the period from July 1964 to December 2009 with $T = 546$. For comparison purposes with a standard methodology for small n, we consider the 25 and 100 Fama-French (FF) portfolios as base assets. We have downloaded the time series of factors, portfolios and portfolio characteristics from the website of Kenneth French.

4.2 Estimation results

We first present unconditional estimates before looking at the path of the time-varying estimates. We use $\chi_{1,T} = 15$ and $\chi_{2,T} = 546/12$ for the unconditional estimation and $\chi_{1,T} = 15$ and $\chi_{2,T} = 546/36$ for the conditional estimation. In the reported results for the four-factors model, we denote by n^χ the dimension of the cross-section after trimming. We use a data-driven threshold selected by cross-validation as in Bickel and

Levina (2008). Table 1 gathers the estimated annual risk premia for the following unconditional models: the four-factor model, the Fama-French model, and the CAPM. In Table 2, we display the estimates of the components of ν. When n is large, we use bias-corrected estimates for λ and ν. When n is small, we use asymptotics for fixed n and $T \to \infty$. The estimated risk premia for the market factor are of the same magnitude and all positive across the three universes of assets and the three models. The 95% confidence intervals are larger by construction for fixed n, and they often contain the interval for large n. For the four-factor model and the individual stocks the size factor is positively remunerated (2.91%) and it is not significantly different from zero. The value factor commands a significant negative reward (-4.55%). Phalippou (2007) obtained a similar result, indeed he got a growth premium when portofolios are built on stocks with a high institutional ownership. The momentum factor is largely remunerated (7.34%) and significantly different from zero. For the 25 and 100 FF portfolios we observe that the size factor is not significantly positively remunerated while the value factor is significantly positively remunerated (4.81% and 5.11%). The momentum factor bears a significant positive reward (34.03% and 17.29%). The large, but imprecise, estimate for the momentum premium when $n = 25$ and $n = 100$ comes from the estimate for ν_{mom} (25.40% and 8.66%) that is much larger and less accurate than the estimates for ν_m, ν_{smb} and ν_{hml} (0.85%, -0.26%, 0.03%, and 0.55%, 0.01%, 0.33%). Moreover, while for portfolios the estimates of ν_m, ν_{smb} and ν_{hml} are statistically not significant, for individual stocks these estimates are statistically different from zero. In particular, the estimate of ν_{hml} is large and negative, which explains the negative estimate on the value premium displayed in Table 1.

As showed in Figure 1, a potential explanation of the discrepancies revealed in Tables 1 and 2 between individual stocks and portfolios is the much larger heterogeneity of the factor loadings for the former. The portfolio betas are all concentrated in the middle of the cross-sectional distribution obtained from the individual stocks. Creating portfolios distorts information by shrinking the dispersion of betas. The estimation results for the momentum factor exemplify the problems related to a small number of portfolios exhibiting a tight factor structure (Lewellen, Nagel and Shanken (2010)). For λ_m, λ_{smb}, and λ_{hml}, we obtain similar inferential results when we consider the Fama-French model. Our point estimates for λ_m, λ_{smb} and λ_{hml}, for large n agree with Ang, Liu and Schwarz (2008). Our point estimates and confidence intervals for λ_m, λ_{smb} and λ_{hml}, agree with the results reported by Shanken and Zhou (2007) for the 25 portfolios.

Figure 2 plots the estimated time-varying path of the four risk premia from the individual stocks. We also plot the unconditional estimates and the average lambda over time. The discrepancy between the unconditional estimate and the average over time is explained by a well-known bias coming from market-timing and volatility-timing (Jagannathan and Wang (1996), Lewellen and Nagel (2006), Boguth, Carlson, Fisher and Simutin (2010)). The risk premia for the market, size and value factors feature a counter-cyclical pattern. Indeed, these risk premia increase during economic contractions and decrease during economic booms. Gomes, Kogan and Zhang (2003) and Zhang (2005) construct equilibrium models exhibiting a countercyclical behavior in size and book-to-market effects. On the contrary, the risk premium for momentum factor is pro-cyclical. Furthermore, conditional estimates of the value premium take stable and positive values. They are not significantly different from zero during economic booms. The conditional estimates of the size premium are most of the time slightly positive, and not significantly different from zero.

Figure 3 plots the estimated time-varying path of the four risk premia from the 25 portfolios. We also plot the unconditional estimates and the average lambda over time. The discrepancy between the unconditional estimate and the averages over time is also observed for $n = 25$. The conditional point estimates for $\lambda_{mom,t}$ are larger and more imprecise than the unconditional estimate in Table 1. Indeed, the pointwise confidence intervals contain the confidence interval of the unconditional estimate for λ_{mom}. Finally, by comparing Figures 2 and 3, we observe that the patterns of risk premia look similar except for the book-to-market factor. Indeed, the risk premium for the value effect estimated from the 25 portfolios is pro-cyclical, contradicting the counter-cyclical behavior predicted by finance theory. By comparing Figures 3 and 4, we observe that increasing the number of portfolios to 100 does not help in reconciling the discrepancy.

4.3 Specification test results

As already mentioned Figure 1 shows that the 25 FF portfolios all have four-factor market and momentum betas close to one and zero, respectively, so the model can be thought as a two-factor model consisting of smb and hml for the purposes of explaining cross-sectional variation in expected returns. For the 100 FF portfolios the dispersion around one and zero is slightly larger. As depicted in Figure 1 by Lewellen, Nagel and Shanken (2010), this empirical concentration implies that it is easy to get artificially large estimates $\hat{\rho}^2$ of the cross-sectional R^2 for three- and four-factor models. On the contrary, the observed heterogeneity in the betas coming from the individual stocks impedes this. This suggests that it is much less easy to find factors that explain the cross-sectional variation of expected returns on individual stocks than on portfolios. Reporting large $\hat{\rho}^2$, or small SSR $\hat{Q}_e$, when n is large, is much more impressive than when n is small.

Table 2 gathers specification test results for unconditional factor models. As already mentioned, when n is large, we prefer working with test statistics based on the SSR $\hat{Q}_e$ instead of $\hat{\rho}^2$ since the population R^2 is not well-defined with tradable factors under the null hypothesis of well-specification (its denominator is zero). For the individual stocks, we compute the test statistic $\tilde{\Sigma}_{\xi}^{-1/2}\hat{\xi}_{nT}$ as well as its associated p-value. For the 25 and 100 FF portfolios, we compute weighted test statistics (Gibbons, Ross and Shanken (1989)) as well as their associated p-value. We do similarly for the test statistics relying on the alphas a. As expected the rejection of the well specification is strong on the individual stocks. This suggests that the unconditional models do not describe the behavior of individual stocks. For the 25 portfolios, the Gibbons-Ross-Shanken test statistic rejects the well specification for the CAPM and the three-factor model. The four-factor model is not rejected at 1% level, but it is rejected at 5% level.

4.4 Cost of equity

The results in Section 3 can be used for estimation and inference on the cost of equity in conditional factor models. We can estimate the time varying cost of equity $CE_{i,t} = r_{f,t} + b_{i,t}'\lambda_t$ of firm i with $\widehat{CE}_{i,t} = r_{f,t} + \hat{b}_{i,t}'\hat{\lambda}_t$, where $r_{f,t}$ is the risk-free rate. We have (see Appendix 3)

$$\begin{aligned} \sqrt{T}\left(\widehat{CE}_{i,t} - CE_{i,t}\right) &= \psi_{i,t}'E_2'\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) \\ &\quad + \left(Z_{t-1}' \otimes b_{i,t}'\right) W_{p,K}\sqrt{T} vec\left[\hat{\Lambda}' - \Lambda'\right] + o_p(1), \end{aligned} \tag{17}$$

where $\psi_{i,t} = \left(\lambda_t' \otimes Z_{t-1}', \lambda_t' \otimes Z_{i,t-1}'\right)'$. Standard results on OLS imply that estimator $\hat{\beta}_i$ is asymptotically normal, $\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) \Rightarrow N\left(0, \tau_i^2 Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1}\right)$, and independent of estimator $\hat{\Lambda}$. Then, from Proposition 7 we deduce that $\sqrt{T}\left(\widehat{CE}_{i,t} - CE_{i,t}\right) \Rightarrow N\left(0, \Sigma_{CE_{i,t}}\right)$, conditionally on Z_{t-1}, where

$$\Sigma_{CE_{i,t}} = \tau_i^2 \psi_{i,t}' E_2' Q_{x,i}^{-1} S_{ii} Q_{x,i}^{-1} E_2 \psi_{i,t} + \left(Z_{t-1}' \otimes b_{i,t}'\right) W_{p,K} \Sigma_{\Lambda} W_{K,p} \left(Z_{t-1} \otimes b_{i,t}\right).$$

Figure 5 plots the path of the estimated annualized costs of equity for Ford Motor, Disney, Motorola and Sony. The cost of equity has risen tremendously during the recent subprime crisis.

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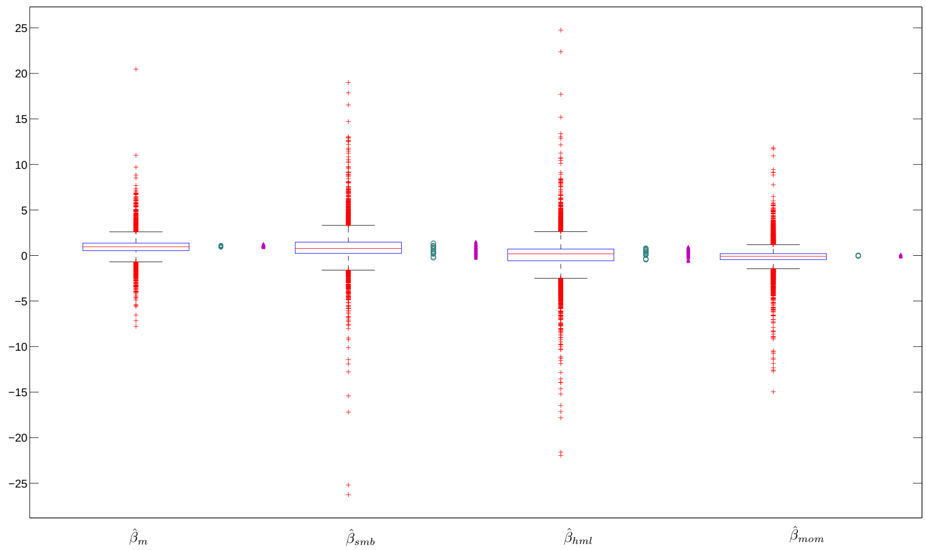
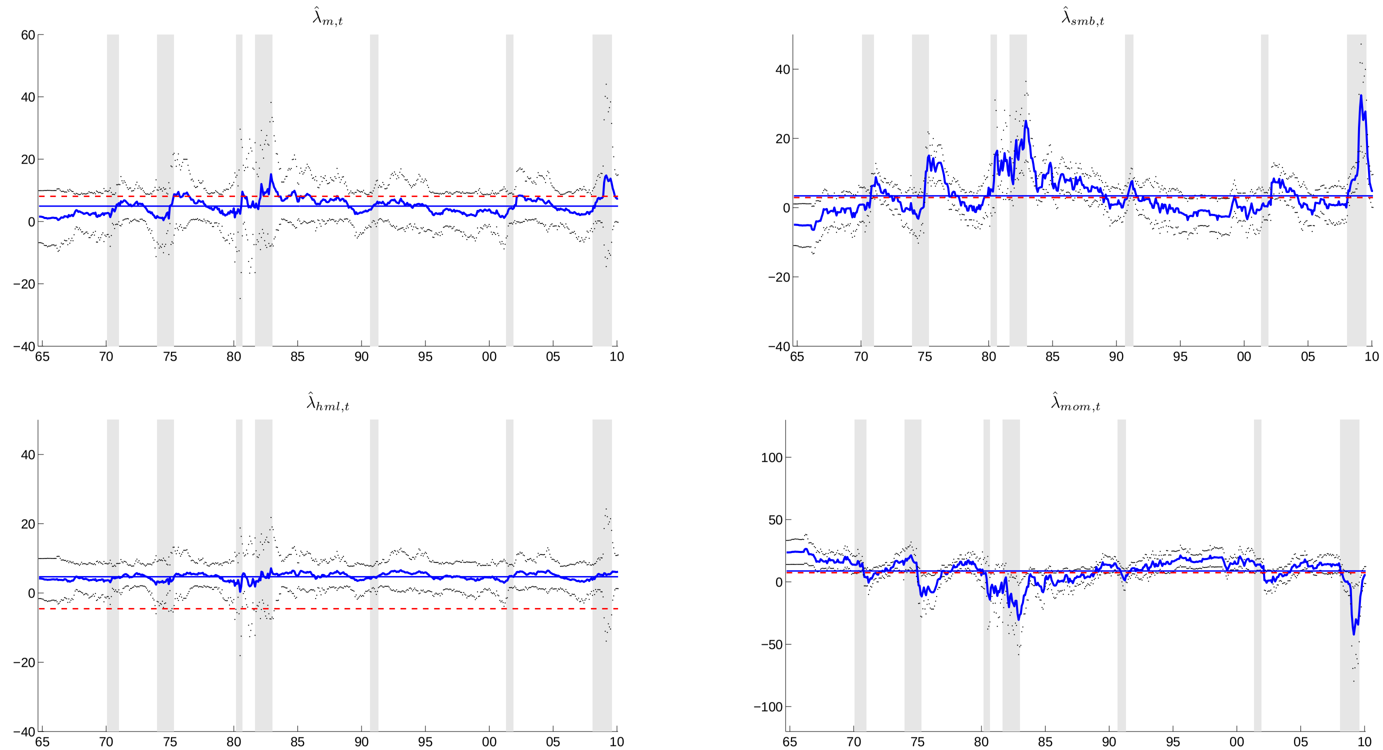
Figure 1: Distribution of the factor loadings



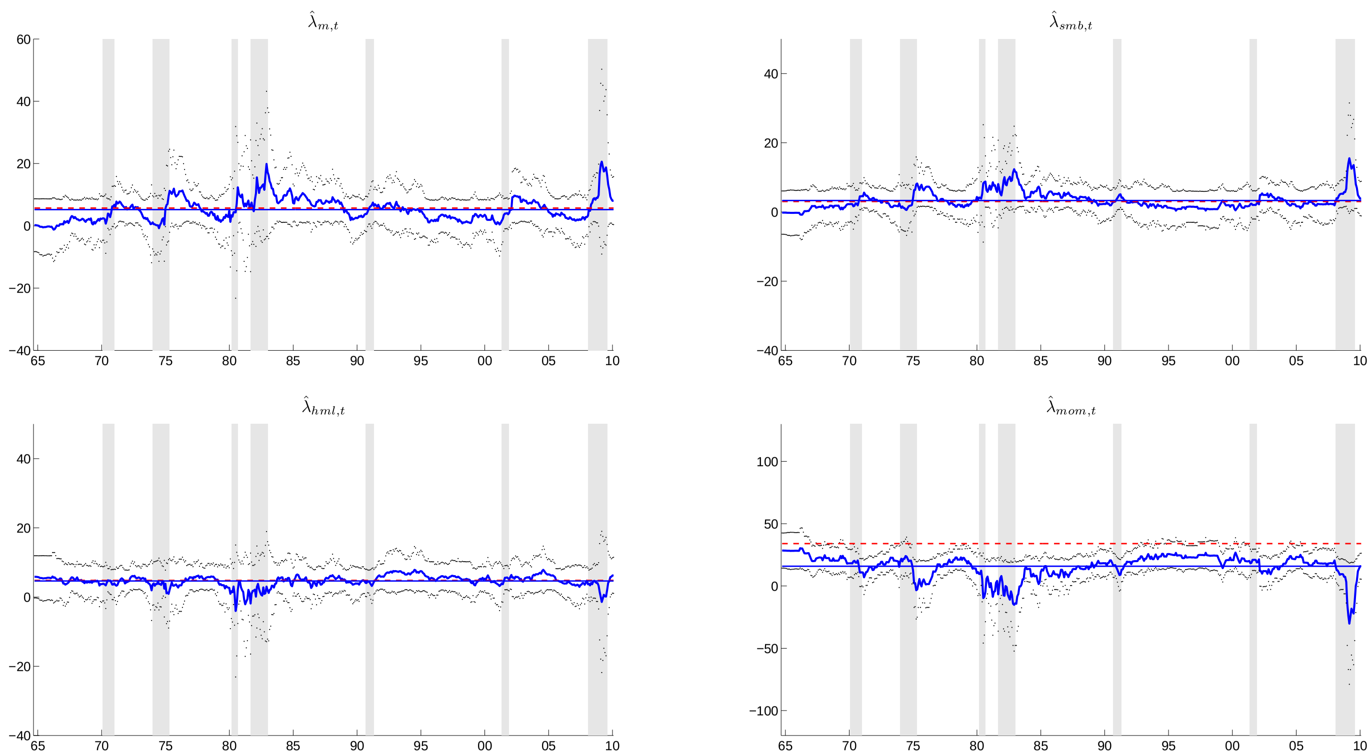
The figure displays box-plots for the distribution of factor loadings $\hat{\beta}_m, \hat{\beta}_{smb}, \hat{\beta}_{hml}$ and $\hat{\beta}_{mom}$. The factor loadings are estimated by running the time-series OLS regression in equation (2) for $n = 9,936$ from 1964/07 to 2009/12. Moreover, next to each box-plot we report the estimated factor loadings for the 25 and 100 Fama-French portfolios (circles and triangles, respectively).

Figure 2: **Path of estimated annualized risk premia with $n = 9,936$**



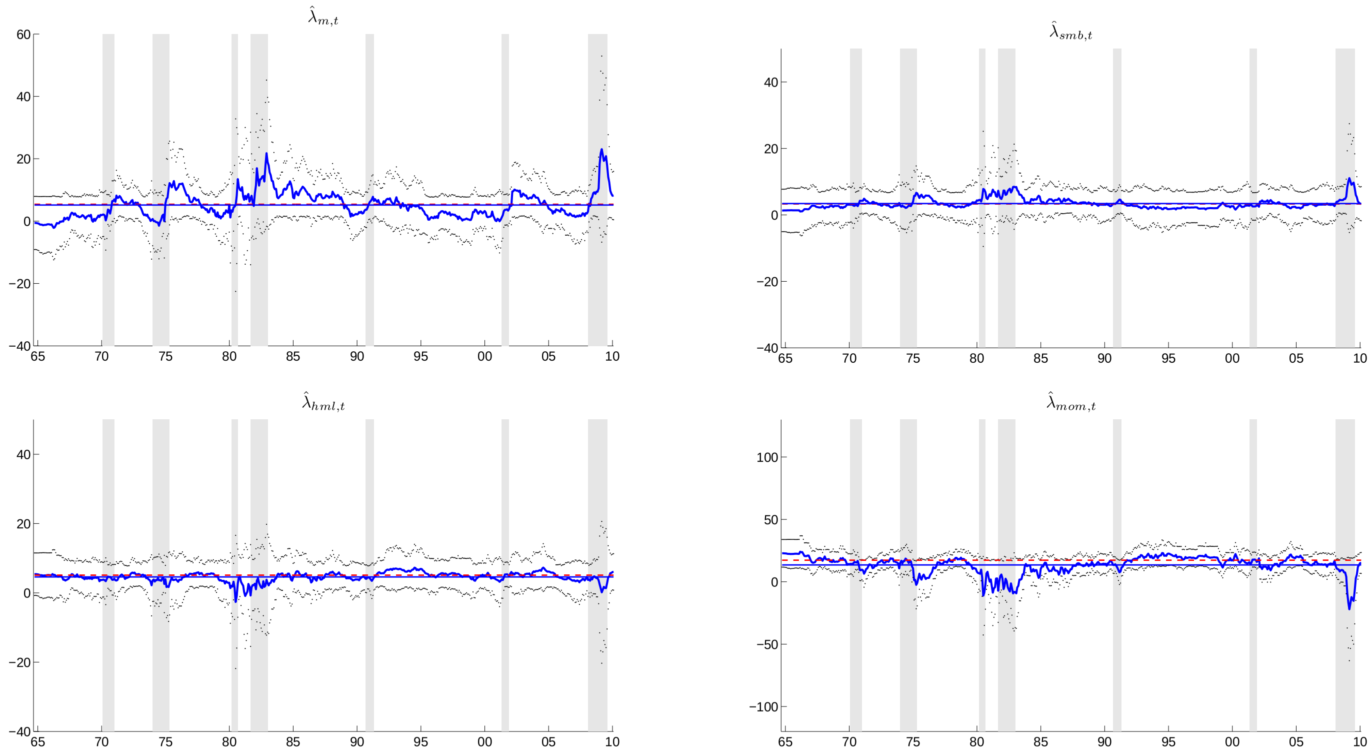
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_m$, $\hat{\lambda}_{smb}$, $\hat{\lambda}_{hml}$ and $\hat{\lambda}_{mom}$ and their pointwise confidence intervals at 95% probability level. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). We consider all stocks ($n = 9,926$ and $n^{\chi} = 2,612$) as base assets. The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER). The recessions start at the peak of a business cycle and end at the trough.

Figure 3: Path of estimated annualized risk premia with $n = 25$



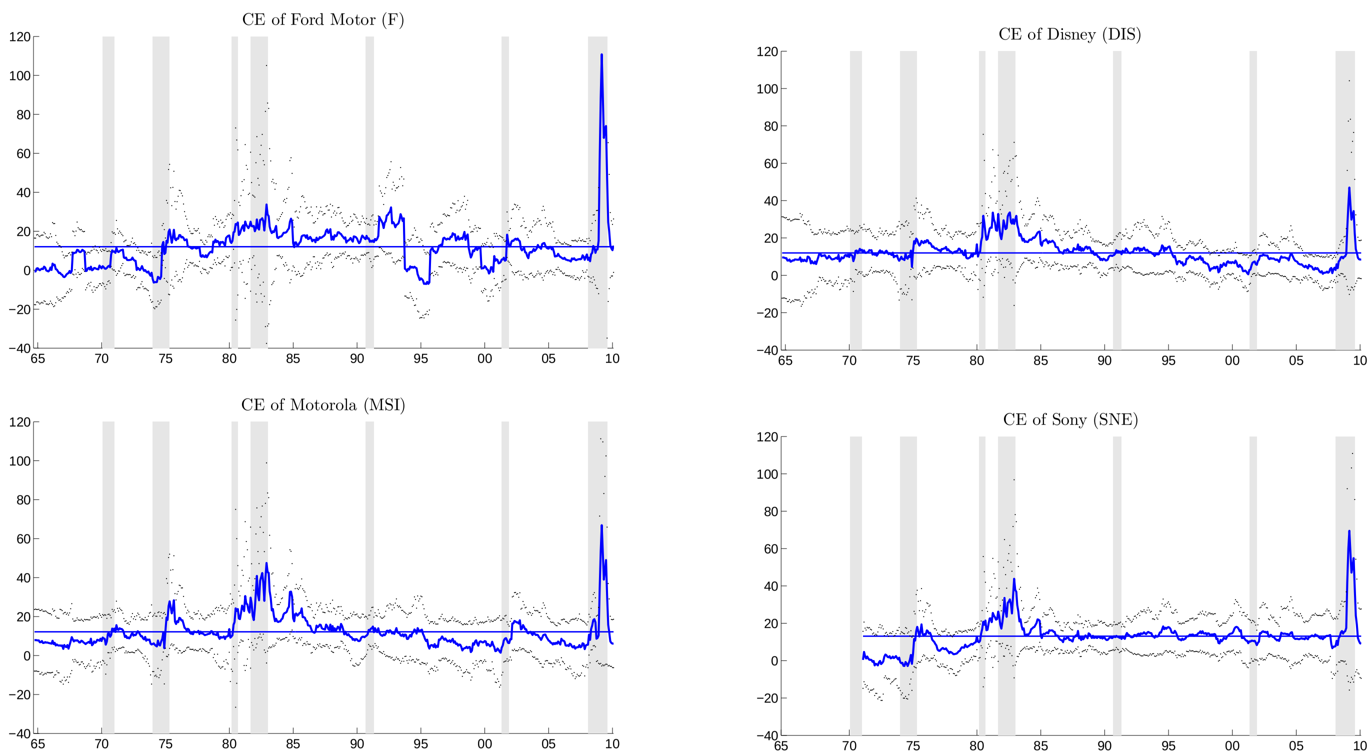
The figure plots the path of estimated annualized risk premia $\hat{\lambda}_m$, $\hat{\lambda}_{smb}$, $\hat{\lambda}_{hml}$ and $\hat{\lambda}_{mom}$ and their pointwise confidence intervals at 95% probability level. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER).

Figure 4: Path of estimated annualized risk premia with $n = 100$



The figure plots the path of estimated annualized risk premia $\hat{\lambda}_m$, $\hat{\lambda}_{smb}$, $\hat{\lambda}_{hml}$ and $\hat{\lambda}_{mom}$ and their pointwise confidence intervals at 95% probability level. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determined by the National Bureau of Economic Research (NBER).

Figure 5: Path of estimated annualized costs of equity



The figure plots the path of estimated annualized cost of equities for Ford Motor, Disney Walt, Motorola and Sony and their pointwise confidence intervals at 95% probability level. We also report the unconditional estimate (dashed horizontal line) and the average conditional estimate (solid horizontal line). The vertical shaded areas denote recessions determinated by the National Bureau of Economic Research (NBER).

Table 1: Estimated annualized risk premia for the unconditional models

	Stocks ($n = 9,936$, $n^{\chi} = 9,902$)		Portfolios ($n = n^{\chi} = 25$)		Portfolios ($n = n^{\chi} = 100$)	
	bias corrected estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval
			Four-factor model			
λ_m	8.08	(3.20, 12.99)	5.70	(0.73, 10.67)	5.41	(0.42, 10.39)
λ_{smb}	2.91	(-0.45, 6.26)	3.02	(-0.48, 6.51)	3.28	(-0.27, 6.83)
λ_{hml}	-4.55	(-8.01, -1.08)	4.81	(1.21, 8.41)	5.11	(1.52, 8.71)
λ_{mom}	7.34	(2.74, 11.94)	34.03	(9.98, 58.07)	17.29	(8.55, 26.03)
			Fama-French model			
λ_m	7.60	(2.72, 12.49)	5.04	(0.11, 9.97)	4.88	(-0.08, 0.83)
λ_{smb}	2.73	(-0.62, 6.09)	3.00	(-0.42, 6.42)	3.35	(-0.13, 6.83)
λ_{hml}	-4.95	(-8.42, -1.49)	5.20	(1.66, 8.74)	5.20	(1.63, 8.77)
			CAPM			
λ_m	7.39	(2.50, 12.27)	6.98	(1.93, 12.02)	7.16	(2.06, 12.25)

The table contains the estimated annualized risk premia for the market (λ_m), size (λ_{smb}), book-to-market (λ_{hml}) and momentum (λ_{mom}) factors. The bias corrected estimates $\hat{\lambda}_B$ of λ are reported for individual stocks ($n = 9,936$). In order to build the confidence intervals for $n = 9,936$, we use $\hat{\Sigma}_f$. When we consider 25 and 100 portfolios as base assets, we compute an estimate of the covariance matrix $\Sigma_{\lambda,n}$ defined in Section 2.3.

Table 2: Estimated annualized ν for the unconditional models

	Stocks ($n = 9,936$, $n^{\chi} = 9,902$)		Portfolios ($n = n^{\chi} = 25$)		Portfolios ($n = n^{\chi} = 100$)	
	bias corrected estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval	point estimate (%)	95% conf. interval
Four-factor model						
ν_m	3.22	(2.95, 3.50)	0.85	(-0.10, 1.79)	0.55	(-0.46, 1.57)
ν_{smb}	-0.37	(-0.67, -0.06)	-0.26	(-1.24, 0.72)	0.01	(-1.14, 1.16)
ν_{hml}	-9.33	(-9.67, -8.90)	0.03	(-0.95, 1.01)	0.33	(-0.63, 1.30)
ν_{mom}	-1.29	(-1.88, -0.70)	25.40	(1.80, 49.00)	8.66	(1.23, 16.10)
Fama-French model						
ν_m	2.75	(2.48, 3.02)	0.18	(-0.51, 0.87)	0.02	(-0.84, 0.88)
ν_{smb}	-0.54	(-0.85, -0.22)	-0.27	(-0.93, 0.40)	0.08	(-0.85, 1.01)
ν_{hml}	-9.74	(-10.08, -9.39)	0.41	(-0.32, 1.15)	0.42	(-0.44, 1.28)
CAPM						
ν_m	2.53	(2.32, 2.74)	2.12	(0.85, 3.40)	2.30	(0.84, 3.77)

The table contains the annualized estimates of the components of vector ν for the market (ν_m), size (ν_{smb}), book-to-market (ν_{hml}) and momentum (ν_{mom}) factors. The bias corrected estimates $\hat{\nu}_B$ of ν are reported for individual stocks ($n = 9,936$). In order to build the confidence intervals, we compute $\hat{\Sigma}_{\nu}$ in Proposition 4 for $n = 9,936$. When we consider 25 and 100 portfolios as base assets, we compute an estimate of the covariance matrix $\Sigma_{\nu,n}$ defined in Section 2.3.

Table 3: Specification test results for the unconditional models

	Test statistic based on $\hat{Q}_e, \mathcal{H}_0: a = b'\nu$			Test statistic based on $\hat{Q}_a, \mathcal{H}_0: a = 0$		
	Stocks ($n = 9,936$)	Portfolios ($n = 25$)	Portfolios ($n = 100$)	Stocks ($n = 9,936$)	Portfolios ($n = 25$)	Portfolios ($n = 100$)
Four-factor model						
Test statistic	22.9551	35.2231	253.2575	43.2804	74.9100	263.3395
p-value	0.0000	0.0267	0.0000	0.0000	0.0000	0.0000
Fama-French model						
Test statistic	20.8816	83.6846	253.9652	40.2845	87.3767	270.7899
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
CAPM						
Test statistic	22.3152	110.8368	276.3679	26.1799	111.6735	278.3949
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

The test statistic $\tilde{\Sigma}_{\xi}^{-1/2}\hat{\xi}_{nT}$ defined in Proposition 5 is computed for $n = 9,936$. For $n = 25$ and $n = 100$, the test statistic $T\hat{e}'\Omega^{-1}\hat{e}$ is reported. The test statistic $T\hat{a}'\Omega^{-1}\hat{a}$ is also computed. The table reports the p-values, respectively.

Appendix 1: Regularity conditions

In this Appendix, we list and comment the additional assumptions used to derive the large sample properties of the estimators and test statistics. For unconditional models, we use Assumptions C.1-C.5 below with $x_t = (1, f_t')'$.

Assumption C.1 *There exists constants $\eta, \bar{\eta} \in (0,1]$ and $C_1, C_2, C_3, C_4 > 0$ such that for all $\delta > 0$ and $T \in \mathbb{N}$ we have:*

a) $\mathbb{P}\left[\left\| \frac{1}{T} \sum_t \left(x_t x_t' - E\left[x_t x_t'\right]\right) \right\| \geq \delta\right] \leq C_1 T \exp\left\{-C_2 \delta^2 T^{\eta}\right\} + C_3 \delta^{-1} \exp\left\{-C_4 T^{\bar{\eta}}\right\}.$

Furthermore, for all $\delta > 0$, $T \in \mathbb{N}$, and $1 \leq k, l, m \leq K+1$, the same upper bound holds for:

b) $\sup_{\gamma \in [0,1]} \mathbb{P}\left[\left\| \frac{1}{T} \sum_t I_t(\gamma) \left(x_t x_t' - E\left[x_t x_t'\right]\right) \right\| \geq \delta\right];$

c) $\sup_{\gamma \in [0,1]} \mathbb{P}\left[\left\| \frac{1}{T} \sum_t I_t(\gamma) x_t \varepsilon_t(\gamma) \right\| \geq \delta\right];$

d) $\sup_{\gamma, \gamma' \in [0,1]} \mathbb{P}\left[\left\| \frac{1}{T} \sum_t I_t(\gamma) I_t(\gamma') \left(\varepsilon_t(\gamma)\varepsilon_t(\gamma') x_t x_t' - E\left[\varepsilon_t(\gamma)\varepsilon_t(\gamma') x_t x_t'\right]\right) \right\| \geq \delta\right];$

e) $\sup_{\gamma, \gamma' \in [0,1]} \mathbb{P}\left[\left| \frac{1}{T} \sum_t I_t(\gamma) I_t(\gamma') x_{k,t} x_{l,t} x_{m,t} \varepsilon_t(\gamma) \right| \geq \delta\right].$

Assumption C.2 *There exists $c > 0$ such that* $\sup_{\gamma \in [0,1]} E\left[\left\| \frac{1}{T} \sum_t I_t(\gamma) \left(x_t x_t' - E[x_t x_t']\right) \right\|^4\right] = O(T^{-c}).$

Assumption C.3 *a) There exists a constant M such that $\|x_t\| \leq M$, P-a.s.. Moreover, b)* $\sup_{\gamma \in [0,1]} \|\beta(\gamma)\| < \infty$ *and c)* $\inf_{\gamma \in [0,1]} E[I_t(\gamma)] > 0.$

Assumption C.4 *There exists a constant M such that for all n, T:*

a) $\frac{1}{nT^2} \sum_{i,j} \sum_{t_1,t_2,t_3,t_4} \left|E\left[\varepsilon_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{j,t_3}\varepsilon_{j,t_4} | \gamma_i, \gamma_j\right]\right| \leq M;$

b) $\frac{1}{nT^2} \sum_{i,j} \sum_{t_1,t_2,t_3,t_4} \left\|E\left[\eta_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{j,t_3}\eta_{j,t_4} | \gamma_i, \gamma_j\right]\right\| \leq M$*, where* $\eta_{i,t} = \varepsilon_{i,t}^2 x_t x_t' - E[\varepsilon_{i,t}^2 x_t x_t' | \gamma_i];$

c) $\frac{1}{nT^3} \sum_{i,j} \sum_{t_1,\ldots,t_6} \left|E\left[\varepsilon_{i,t_1}\varepsilon_{i,t_2}\varepsilon_{i,t_3}\varepsilon_{j,t_4}\varepsilon_{j,t_5}\varepsilon_{j,t_6} | \gamma_i, \gamma_j\right]\right| \leq M;$

Assumption C.5 *The trimming constants satisfy $\chi_{1,T} = O\left((\log T)^{\kappa_1}\right)$, $\chi_{2,T} = O\left((\log T)^{\kappa_2}\right)$, with $\kappa_1, \kappa_2 > 0$.*

For conditional models, Assumptions C.1-C.5 are used with x_t replaced by $x_{i,t}$ as defined in Section 3.1. More precisely, for Assumptions C.1a) and C.3a) we replace x_t by $x_t(\gamma)$ and require the bound to be valid uniformly w.r.t. $\gamma \in [0, 1]$; for Assumptions C.1b)-e) and C.2 we replace x_t by $x_t(\gamma)$; for Assumption C.4b) we replace x_t by $x_{i,t}$. Furthermore, we use:

Assumption C.6 *There exists a constant M such that $\left\| E\left[u_t u_t' | Z_{t-1}\right] \right\| \leq M$ for all t, where $u_t = f_t - FZ_{t-1}$.*

Appendix 2: Unconditional factor model

A.2.1 Proof of Proposition 1

To ease notations, we assume w.l.o.g. that the continuous distribution G is uniform on $[0,1]$. For a given countable collection of assets $\gamma_1, \gamma_2, ...$ in $[0,1]$, let $\mu_n = A_n + B_n E[f_1|\mathcal{F}_0]$ and $\Sigma_n = B_n V[f_1|\mathcal{F}_0] B_n'$ $+\Sigma_{\varepsilon,1,n}$, for $n \in \mathbb{N}$, be the mean vector and the covariance matrix of asset excess returns $(R_1(\gamma_1), ..., R_1(\gamma_n))'$ conditional on $\mathcal{F}_0$, where $A_n = [a(\gamma_1), ..., a(\gamma_n)]'$, and $B_n = [b(\gamma_1), ..., b(\gamma_n)]'$. Let $e_n = \mu_n - B_n \left(B_n' B_n\right)^{-1} B_n' \mu_n = A_n - B_n \left(B_n' B_n\right)^{-1} B_n' A_n$ be the residual of the orthogonal projection of μ_n (and A_n) onto the columns of B_n. Furthermore, let $\mathcal{P}_n$ denote the set of static portfolios p_n that invest in the risk-free asset and risky assets $\gamma_1, ..., \gamma_n$, for $n \in \mathbb{N}$, with portfolio shares independent of $\mathcal{F}_0$, and let $\mathcal{P}$ denote the set of portfolio sequences (p_n), with $p_n \in \mathcal{P}_n$. For portfolio $p_n \in \mathcal{P}_n$, the cost, the conditional expected return, and the conditional variance are given by $C(p_n) = \alpha_{0,n} + \alpha_n' \iota_n$, $E[p_n|\mathcal{F}_0] = R_0 C(p_n) + \alpha_n' \mu_n$, and $V[p_n|\mathcal{F}_0] = \alpha_n' \Sigma_n \alpha_n$, where $\iota_n = (1, ..., 1)'$ and $\alpha_n = (\alpha_{1,n}, ..., \alpha_{n,n})'$. Moreover, let $\rho = \sup_p E[p|\mathcal{F}_0]/V[p|\mathcal{F}_0]^{1/2}$ s.t. $p \in \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, with $C(p) = 0$ and $p \neq 0$, be the maximal Sharpe ratio of zero-cost portfolios. For expository purpose, we do not make explicit the dependence of μ_n, Σ_n, e_n, $\mathcal{P}_n$, and ρ on the collection of assets (γ_i).

The statement of Proposition 1 is proved by contradiction. Suppose that $\inf_{\nu\in\mathbb{R}^K} \int [a(\gamma) - b(\gamma)'\nu]^2 d\gamma = \int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma > 0$, where $\nu_\infty = \left(\int b(\gamma) b(\gamma)' d\gamma\right)^{-1} \int b(\gamma) a(\gamma) d\gamma$. By the strong LLN and Assumption APR.2, we have that:

$$\frac{1}{n}\|e_n\|^2 = \inf_{\nu\in\mathbb{R}^K} \frac{1}{n}\sum_{i=1}^{n}[a(\gamma_i) - b(\gamma_i)'\nu]^2 \to \int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma, \tag{18}$$

as $n \to \infty$, for any sequence (γ_i) in a set $\mathcal{J}_1 \subset \Gamma$, with measure $\mu_\Gamma(\mathcal{J}_1) = 1$. Let us now show that an asymptotic arbitrage portfolio exists based on any sequence in $\mathcal{J}_1$ such that $eig_{\max}(\Sigma_{\varepsilon,1,n}) = o(n)$ (Assumption APR.4 (i)). Define the portfolio sequence (q_n) with investments $\alpha_n = \frac{1}{\|e_n\|^2} e_n$ and $\alpha_{0,n} = -\iota_n' \alpha_n$. This static portfolio has zero cost, i.e., $C(q_n) = 0$, while $E[q_n|\mathcal{F}_0] = 1$ and $V[q_n|\mathcal{F}_0] \leq eig_{\max}(\Sigma_{\varepsilon,1,n})\|e_n\|^{-2}$. Moreover, we have $V[q_n|\mathcal{F}_0] = E\left[(q_n - E[q_n|\mathcal{F}_0])^2 |\mathcal{F}_0\right] \geq E\left[(q_n - E[q_n|\mathcal{F}_0])^2 |\mathcal{F}_0, q_n \leq 0\right] P[q_n \leq 0|\mathcal{F}_0] \geq P[q_n \leq 0|\mathcal{F}_0]$. Hence, we get: $P[q_n > 0|\mathcal{F}_0] \geq 1 - V[q_n|\mathcal{F}_0] \geq 1 - eig_{\max}(\Sigma_{\varepsilon,1,n})\|e_n\|^{-2}$. Thus, from $eig_{\max}(\Sigma_{\varepsilon,1,n}) = o(n)$ and $\|e_n\|^{-2} = O(1/n)$, we get $P[q_n > 0|\mathcal{F}_0] \to 1$, P-a.s.. By using the Law of Iterated Expectation and the Lebesgue dominated convergence theorem, $P[q_n > 0] \to 1$. Hence, portfolio (q_n) is an asymptotic arbitrage opportunity. Since asymptotic arbitrage portfolios are ruled out by Assumption APR.5, it follows that we must have $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma = 0$, that is, $a(\gamma) = b(\gamma)'\nu$, for $\nu = \nu_\infty$ and almost all $\gamma \in [0,1]$. Such vector ν is unique by Assumption APR.2.

Let us now establish the link between the no-arbitrage conditions and asset pricing restrictions in CR on the one hand, and the asset pricing restriction (3) in the other hand. Let $\mathcal{J} \subset \Gamma$ denote the set of countable collections of assets (γ_i) such that the two conditions: (i) If $V[p_n|\mathcal{F}_0] \to 0$ and $C(p_n) \to 0$, then $E[p_n|\mathcal{F}_0] \to 0$, (ii) If $V[p_n|\mathcal{F}_0] \to 0$, $C(p_n) \to 1$ and $E[p_n|\mathcal{F}_0] \to \delta$, then $\delta \geq 0$, hold for any static portfolio sequence (p_n) in $\mathcal{P}$, P-a.s.. Condition (i) means that, if the conditional variability and cost vanish, so does the conditional expected return. Condition (ii) means that, if the conditional variability vanishes and the cost is positive, the conditional expected return is non-negative. They correspond to Conditions A.1 (i) and (ii) in CR written conditionally on $\mathcal{F}_0$ and for a given countable collection of assets (γ_i). Hence, the set $\mathcal{J}$ is the set permitting no asymptotic arbitrage opportunities in the sense of CR.

Proposition APR: *Under Assumptions APR.1-APR.4, either* $\mu_\Gamma\left(\inf_{\nu\in\mathbb{R}^K}\sum_{i=1}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) = \mu_\Gamma(\mathcal{J}) = 1$*, or* $\mu_\Gamma\left(\inf_{\nu\in\mathbb{R}^K}\sum_{i=1}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) = \mu_\Gamma(\mathcal{J}) = 0$*. The former case occurs if, and only if, the asset pricing restriction (3) holds.*

The fact that $\mu_\Gamma\left(\inf_{\nu\in\mathbb{R}^K}\sum_{i=1}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right)$ is either $= 1$, or $= 0$, is a consequence of the Kolmogorov zero-one law (e.g., Billingsley (1995)). Indeed, $\inf_{\nu\in\mathbb{R}^K}\sum_{i=1}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty$ if, and only if, $\inf_{\nu\in\mathbb{R}^K}\sum_{i=n}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty$, for any $n \in \mathbb{N}$. Thus, the law applies since the event $\inf_{\nu\in\mathbb{R}^K}\sum_{i=1}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty$ belongs to the tail sigma-field $\mathcal{T} = \bigcap_{n=1}^{\infty}\sigma(\gamma_i, i = n, n+1, ...)$, and the variables γ_i are i.i.d. under μ_Γ.

Proof of Proposition APR: The proof involves four steps.

STEP 1: If $\mu_\Gamma\left(\inf_{\nu\in\mathbb{R}^K}\sum_{i=1}^{\infty}[a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty\right) > 0$, then the asset pricing restriction (3) holds. This step is proved by contradiction. Suppose that the asset pricing restriction (3) does not hold, and thus

$\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma > 0$. Then, we get $\mu_\Gamma \left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty \right) = 0$, by the convergence in (18).

STEP 2: If the asset pricing restriction (3) holds, then $\mu_\Gamma \left(\inf_{\nu \in \mathbb{R}^K} \sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu]^2 < \infty \right) = 1$. Indeed, $\mu_\Gamma \left(\sum_{i=1}^{\infty} [a(\gamma_i) - b(\gamma_i)'\nu^*]^2 = 0 \right) = 1$, if the asset pricing restriction (3) holds for some vector $\nu^* \in \mathbb{R}^K$.

STEP 3: If $\mu_\Gamma(\mathcal{J}) > 0$, then the asset pricing restriction (3) holds. By following the same arguments as in CR on p. 1295-1296, we have $\rho^2 \geq \mu_n' \Sigma_{\varepsilon,1,n}^{-1} \mu_n$ and $\Sigma_{\varepsilon,1,n}^{-1} \geq eig_{\max}(\Sigma_{\varepsilon,1,n})^{-1} [I_n - B_n(B_n' B_n)^{-1} B_n']$, for any (γ_i) in $\mathcal{J}$. Thus, we get: $\rho^2 eig_{\max}(\Sigma_{\varepsilon,1,n}) \geq \mu_n' \left(I_n - B_n(B_n' B_n)^{-1} B_n' \right) \mu_n = \min_{\lambda \in \mathbb{R}^K} \| \mu_n - B_n \lambda \|^2 = \min_{\nu \in \mathbb{R}^K} \| A_n - B_n \nu \|^2 = \min_{\nu \in \mathbb{R}^K} \sum_{i=1}^{n} [a(\gamma_i) - b(\gamma_i)'\nu]^2$, for any $n \in \mathbb{N}$, P-a.s.. Hence, we deduce

$$\min_{\nu \in \mathbb{R}^K} \frac{1}{n} \sum_{i=1}^{n} [a(\gamma_i) - b(\gamma_i)'\nu]^2 \leq \rho^2 \frac{1}{n} eig_{\max}(\Sigma_{\varepsilon,1,n}), \tag{19}$$

for any n, P-a.s., and for any sequence (γ_i) in $\mathcal{J}$. Moreover, $\rho < \infty$, P-a.s., by the same arguments as in CR, Corollary 1, and by using that the condition in CR, footnote 6, is implied by our Assumption APR.4 (ii). Then, by the strong LLN, the LHS of Inequality (19) converges to $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma$, for μ_Γ-almost every sequence (γ_i) in $\mathcal{J}$. From Assumption APR.4 (i), the RHS is $o(1)$, P-a.s., for μ_Γ-almost every sequence (γ_i) in Γ. Thus, it follows that $\int [a(\gamma) - b(\gamma)'\nu_\infty]^2 d\gamma = 0$, i.e., $a(\gamma) = b(\gamma)'\nu$, for $\nu = \nu_\infty$ and almost all $\gamma \in [0, 1]$.

STEP 4: If the asset pricing restriction (3) holds, then $\mu_\Gamma(\mathcal{J}) = 1$. If (3) holds, it follows that $e_n = 0$ and $\mu_n = B_n(B_n' B_n)^{-1} B_n' \mu_n$, for all n, for μ_Γ-almost all sequences (γ_i). Then, we get $E[p_n | \mathcal{F}_0] = R_0 C(p_n)$ $+ \alpha_n' B_n (B_n' B_n / n)^{-1} B_n' \mu_n / n$. Moreover, we have: $V[p_n | \mathcal{F}_0] = (B_n' \alpha_n)' V[f_1 | \mathcal{F}_0] (B_n' \alpha_n) + \alpha_n' \Sigma_{\varepsilon,1,n} \alpha_n \geq eig_{\min}(V[f_1 | \mathcal{F}_0]) \left\| B_n' \alpha_n \right\|^2$, where $eig_{\min}(V[f_1 | \mathcal{F}_0]) > 0$, P-a.s. (Assumption APR.4 (iii)). Since $B_n' B_n / n$ converges to a positive definite matrix and $B_n' \mu_n / n$ is bounded, for μ_Γ-almost any sequence (γ_i), Conditions (i) and (ii) in the definition of set $\mathcal{J}$ follow, for μ_Γ-almost any sequence (γ_i), that is, $\mu_\Gamma(\mathcal{J}) = 1$.

A.2.2 Proof of Proposition 2

a) Consistency of $\hat{\nu}$. From Equation (5) and the asset pricing restriction (3), we have:

$$\begin{aligned}\hat{\nu}-\nu &= \hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\hat{b}_i c_\nu'\left(\hat{\beta}_i-\beta_i\right) \quad (20)\\ &= Q_b^{-1}\frac{1}{n}\sum_i \hat{w}_i b_i c_\nu'\left(\hat{\beta}_i-\beta_i\right)+\left(\hat{Q}_b^{-1}-Q_b^{-1}\right)\frac{1}{n}\sum_i \hat{w}_i b_i c_\nu'\left(\hat{\beta}_i-\beta_i\right)\\ &\quad +\hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\left(\hat{b}_i-b_i\right)c_\nu'\left(\hat{\beta}_i-\beta_i\right).\end{aligned}$$

By using $\hat{\beta}_i-\beta_i=\frac{\tau_{i,T}}{\sqrt{T}}\hat{Q}_{x,i}^{-1}Y_{i,T}$ and $\hat{Q}_b^{-1}-Q_b^{-1}=-\hat{Q}_b^{-1}\left(\hat{Q}_b-Q_b\right)Q_b^{-1}$, we get:

$$\begin{aligned}\hat{\nu}-\nu &= \frac{1}{\sqrt{nT}}Q_b^{-1}\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}b_i c_\nu'\hat{Q}_{x,i}^{-1}Y_{i,T}-\frac{1}{\sqrt{nT}}\hat{Q}_b^{-1}\left(\hat{Q}_b-Q_b\right)Q_b^{-1}\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}b_i c_\nu'\hat{Q}_{x,i}^{-1}Y_{i,T}\\ &\quad +\frac{1}{T}\hat{Q}_b^{-1}\frac{1}{n}\sum_i \hat{w}_i\tau_{i,T}^2E_2'\hat{Q}_{x,i}^{-1}Y_{i,T}Y_{i,T}'\hat{Q}_{x,i}^{-1}c_\nu\\ &=: \frac{1}{\sqrt{nT}}Q_b^{-1}I_1-\frac{1}{\sqrt{nT}}\hat{Q}_b^{-1}\left(\hat{Q}_b-Q_b\right)Q_b^{-1}I_1+\frac{1}{T}\hat{Q}_b^{-1}I_2. \quad (21)\end{aligned}$$

To control I_1, we use the decomposition:

$$I_1 = \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}b_i c_\nu'\hat{Q}_x^{-1}Y_{i,T}+\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}b_i c_\nu'\left(\hat{Q}_{x,i}^{-1}-\hat{Q}_x^{-1}\right)Y_{i,T} \qquad =: I_{11}+I_{12}.$$

Write $I_{11}=I_{111}\hat{Q}_x^{-1}c_\nu$ and decompose $I_{111}:=\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}b_iY_{i,T}'$ as:

$$\begin{aligned}I_{111} &= \frac{1}{\sqrt{n}}\sum_i w_i\tau_i b_iY_{i,T}'+\frac{1}{\sqrt{n}}\sum_i\left(\mathbf{1}_i^\chi-1\right)w_i\tau_i b_iY_{i,T}'+\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi w_i\left(\tau_{i,T}-\tau_i\right)b_iY_{i,T}'\\ &\quad +\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi\left(\hat{v}_i^{-1}-v_i^{-1}\right)\tau_{i,T}b_iY_{i,T}' \qquad =: I_{1111}+I_{1112}+I_{1113}+I_{1114}.\end{aligned}$$

We have $E\left[\|I_{1111}\|^2\,|x_{\underline{T}},I_{\underline{T}},\{\gamma_i\}\right]=\frac{1}{nT}\sum_{i,j}\sum_t w_iw_j\tau_i\tau_jI_{i,t}I_{j,t}\sigma_{ij,t}\|x_t\|^2 b_j'b_i$ by Assumption A.1 a). Then, by using $\|x_t\|\leq M$, $\|b_i\|\leq M$, $\tau_i\leq M$, $w_i\leq M$ from Assumption C.3, and Assumption A.1 c), we get $E\left[\|I_{1111}\|^2\,|\{\gamma_i\}\right]\leq C$. Then $I_{1111}=O_p(1)$. To control I_{1112}, we use the next Lemma.

Lemma 1 *Under Assumption C.2:* $\sup_i \mathbb{P}\left[\mathbf{1}_i^\chi=0\right]=O(T^{-\bar{b}})$, *for any* $\bar{b}>0$.

By using $\|I_{1112}\| \leq \frac{C}{\sqrt{n}} \sum_i (1 - \mathbf{1}_i^{\chi}) \|Y_{i,T}\|$, $\sup_i E[\|Y_{i,T}\| | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}] \leq C$ from Assumption A.1, and Lemma 1, it follows $I_{1112} = O_p(\sqrt{n} T^{-\bar{b}})$, for any $\bar{b} > 0$. Since $n \asymp T^{\bar{\gamma}}$, we get $I_{1112} = o_p(1)$. We have $E\left[\|I_{1113}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \leq \frac{C}{nT} \sum_{i,j} \sum_t \mathbf{1}_i^{\chi} \mathbf{1}_j^{\chi} |\tau_{i,T} - \tau_i| |\tau_{j,T} - \tau_j| \sigma_{ij,t}$. Then, by the Cauchy-Schwartz inequality and Assumption A.1 c), we get $E\left[\|I_{1113}\|^2 | \{\gamma_i\}\right] \leq CM \sup_{\gamma \in [0,1]} E\left[\mathbf{1}_i^{\chi} |\tau_{i,T} - \tau_i|^4 | \gamma_i = \gamma\right]^{1/2}$. By using $\tau_{i,T} - \tau_i = -\tau_{i,T} \tau_i \frac{1}{T} \sum_t (I_{i,t} - E[I_{i,t} | \gamma_i])$ we get $\sup_{\gamma \in [0,1]} E\left[\mathbf{1}_i^{\chi} |\tau_{i,T} - \tau_i|^4 | \gamma_i = \gamma\right]$ $\leq C \chi_{2,T}^4 \sup_{\gamma \in [0,1]} E\left[\left|\frac{1}{T} \sum_t (I_t(\gamma) - E[I_t(\gamma)])\right|^4\right] = o(1)$ from Assumptions C.2 and C.5. Then $I_{1113} = o_p(1)$. From $\hat{v}_i^{-1} - v_i^{-1} = -v_i^{-2} (\hat{v}_i - v_i) + \hat{v}_i^{-1} v_i^{-2} (\hat{v}_i - v_i)^2$, we get:

$$
\begin{aligned}
I_{1114} &= -\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} (\hat{v}_i - v_i) \tau_{i,T} b_i Y_{i,T}' + \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} \hat{v}_i^{-1} v_i^{-2} (\hat{v}_i - v_i)^2 \tau_{i,T} b_i Y_{i,T}' \\
&=: I_{11141} + I_{11142}.
\end{aligned}
$$

Let us first consider I_{11141}. We have:

$$
\begin{aligned}
\hat{v}_i - v_i &= \tau_{i,T} c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} \left(\hat{S}_{ii} - S_{ii}\right) \hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1} + 2\tau_{i,T} (c_{\hat{\nu}_1} - c_\nu)' \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1} \\
&\quad + \tau_{i,T} (c_{\hat{\nu}_1} - c_\nu)' \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} (c_{\hat{\nu}_1} - c_\nu) + 2\tau_{i,T} c_\nu' \left(\hat{Q}_{x,i}^{-1} - Q_x^{-1}\right) S_{ii} \hat{Q}_{x,i}^{-1} c_\nu \\
&\quad + \tau_{i,T} c_\nu' \left(\hat{Q}_{x,i}^{-1} - Q_x^{-1}\right) S_{ii} \left(\hat{Q}_{x,i}^{-1} - Q_x^{-1}\right) c_\nu + (\tau_{i,T} - \tau_i) c_\nu' Q_x^{-1} S_{ii} Q_x^{-1} c_\nu,
\end{aligned}
$$

and we get for the first two terms:

$$
\begin{aligned}
I_{111411} &= -\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^2 c_{\hat{\nu}_1}' \hat{Q}_{x,i}^{-1} \left(\hat{S}_{ii} - S_{ii}\right) \hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1} b_i Y_{i,T}', \\
I_{111412} &= -\frac{2}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^2 (c_{\hat{\nu}_1} - c_\nu)' \hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}_1} b_i Y_{i,T}'.
\end{aligned}
$$

We first show $I_{111412} = o_p(1)$. For this purpose, it is enough to show that $c_{\hat{\nu}_1} - c_\nu = O_p(T^{-c})$, for some $c > 0$, and $\frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^2 \left(\hat{Q}_{x,i}^{-1} S_{ii} \hat{Q}_{x,i}^{-1}\right)_{kl} b_i Y_{i,T}' = O_p\left(\chi_{1,T}^2 \chi_{2,T}^2\right)$, for any k, l. The first statement is implied by arguments showing consistency without estimated weights. The second statement follows from $\mathbf{1}_i^{\chi} \|\hat{Q}_{x,i}^{-1}\| \leq C \chi_{1,T}$, $\mathbf{1}_i^{\chi} \tau_{i,T} \leq \chi_{2,T}$ (see control of I_{12} below), and an argument as for I_{1111}. Let us now prove that $I_{111411} = o_p(1)$. For this purpose, it is enough to show that

$$
J_1 := \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^2 \left(\hat{Q}_{x,i}^{-1} \left(\hat{S}_{ii} - S_{ii}\right) \hat{Q}_{x,i}^{-1}\right)_{kl} b_i Y_{i,T}' = o_p(1),
$$

for any k, l. By using $\hat{\varepsilon}_{i,t} = \varepsilon_{i,t} - x_t' \left(\hat{\beta}_i - \beta_i\right) = \varepsilon_{i,t} - \frac{\tau_{i,T}}{\sqrt{T}} x_t' \hat{Q}_{x,i}^{-1} Y_{i,T}$, we get:

$$\begin{aligned} \hat{S}_{ii} - S_{ii} &= \frac{1}{T_i} \sum_t I_{i,t} \left(\hat{\varepsilon}_{i,t}^2 - \varepsilon_{it}^2\right) x_t x_t' + \frac{1}{T_i} \sum_t I_{i,t} \left(\varepsilon_{it}^2 x_t x_t' - S_{ii}\right) \\ &= \frac{\tau_{i,T}}{\sqrt{T}} W_{1,i,T} - \frac{2\tau_{i,T}^2}{T} W_{2,i,T} \hat{Q}_{x,i}^{-1} Y_{i,T} + \frac{\tau_{i,T}^3}{T} \hat{Q}_{x,i}^{(4)} \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1}, \end{aligned}$$

where $W_{1,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} \eta_{i,t}$, $\eta_{i,t} = \varepsilon_{it}^2 x_t x_t' - E\left[\varepsilon_{it}^2 x_t x_t' | \gamma_i\right]$, $W_{2,i,T} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} \varepsilon_{i,t} x_t^3$, $\hat{Q}_{x,i}^{(4)} := \frac{1}{\sqrt{T}} \sum_t I_{i,t} x_t^4$ and x_t has been treated as a scalar to ease notation. Then:

$$\begin{aligned} J_1 &= \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^3 \hat{Q}_{x,i}^{-1} W_{1,i,T} \hat{Q}_{x,i}^{-1} b_i Y_{i,T}' - \frac{2}{\sqrt{nT}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^4 \hat{Q}_{x,i}^{-1} W_{2,i,T} \hat{Q}_{x,i}^{-1} Y_{i,T} \hat{Q}_{x,i}^{-1} b_i Y_{i,T}' \\ &\quad + \frac{1}{\sqrt{nT}} \sum_i \mathbf{1}_i^{\chi} v_i^{-2} \tau_{i,T}^5 \hat{Q}_{x,i}^{-1} \hat{Q}_{x,i}^{(4)} \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-2} b_i Y_{i,T}' \qquad =: J_{11} + J_{12} + J_{13}. \end{aligned}$$

Let us consider J_{11}. We have:

$$E\left[\|J_{11}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \leq \frac{C}{nT^3} \sum_{i,j} \sum_{t_1,t_2,t_3,t_4} \mathbf{1}_i^{\chi} \mathbf{1}_j^{\chi} \tau_{i,T}^3 \tau_{j,T}^3 \|\hat{Q}_{x,i}^{-1}\|^2 \|\hat{Q}_{x,j}^{-1}\|^2 \|E\left[\eta_{i,t_1} \varepsilon_{i,t_2} \varepsilon_{j,t_3} \eta_{j,t_4} | x_{\underline{T}}, \gamma_i, \gamma_j\right]\|.$$

By using $\mathbf{1}_i^{\chi} \|\hat{Q}_{x,i}^{-1}\| \leq C\chi_{1,T}$, $\mathbf{1}_i^{\chi} \tau_{i,T} \leq \chi_{2,T}$, the Law of Iterated Expectations and Assumptions C.4 b) and C.5, we get $E\left[\|J_{11}\|^2 |\right] = o(1)$. Thus $J_{11} = o_p(1)$. By similar argument and using Assumptions C.4 a), c), we get $J_{12} = o_p(1)$ and $J_{13} = o_p(1)$. Hence $J_1 = o_p(1)$. Paralleling the detailed arguments provided above, we can show that all other remaining terms making I_{1114} are also $o_p(1)$. Hence $I_{11} = O_p(1)$.

To control I_{12}, we have:

$$\begin{aligned} I_{12} &= \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} v_i^{-1} \tau_{i,T} b_i c_\nu' \left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) Y_{i,T} \\ &\quad + \frac{1}{\sqrt{n}} \sum_i \mathbf{1}_i^{\chi} \left(\hat{v}_i^{-1} - v_i^{-1}\right) \tau_{i,T} b_i c_\nu' \left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) Y_{i,T} \qquad =: I_{121} + I_{122}. \end{aligned}$$

From $\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1} = -\hat{Q}_x^{-1}\left(\frac{1}{T_i}\sum_t I_{i,t}x_t x_t' - \hat{Q}_x\right)\hat{Q}_{x,i}^{-1} = -\tau_{i,T}\hat{Q}_x^{-1}W_{i,T}\hat{Q}_{x,i}^{-1} + \hat{Q}_x^{-1}W_T\hat{Q}_{x,i}^{-1}$, where $W_{i,T} = \frac{1}{T}\sum_t I_{i,t}(x_t x_t' - Q_x)$ and $W_T = \frac{1}{T}\sum_t (x_t x_t' - Q_x)$, we can write:

$$
\begin{aligned}
I_{121} &= -\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi v_i^{-1}\tau_{i,T}^2 b_i c_\nu' \hat{Q}_x^{-1}W_{i,T}\hat{Q}_{x,i}^{-1}Y_{i,T} + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi v_i^{-1}\tau_{i,T} b_i c_\nu' \hat{Q}_x^{-1}W_T\hat{Q}_{x,i}^{-1}Y_{i,T} \\
&= \left(-\frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi v_i^{-1}\tau_{i,T}^2 b_i Y_{i,T}'\hat{Q}_{x,i}^{-1}W_{i,T} + \frac{1}{\sqrt{n}}\sum_i \mathbf{1}_i^\chi v_i^{-1}\tau_{i,T} b_i Y_{i,T}'\hat{Q}_{x,i}^{-1}W_T\right)\hat{Q}_x^{-1}c_\nu \\
&=: (I_{1211} + I_{1212})\,\hat{Q}_x^{-1}c_\nu.
\end{aligned}
$$

Let us consider term I_{1211}. From Assumption C.3 we have:

$$
E\left[\|I_{1211}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \le \frac{C\chi_{2,T}^4}{nT}\sum_{i,j}\sum_t \mathbf{1}_i^\chi \mathbf{1}_j^\chi |\sigma_{ij,t}|\|\hat{Q}_{x,i}^{-1}\|\|\hat{Q}_{x,j}^{-1}\|\|W_{i,T}\|\|W_{j,T}\|.
$$

Now, by using that $\|\hat{Q}_{x,i}^{-1}\|^2 = Tr\left(\hat{Q}_{x,i}^{-2}\right) = \sum_{k=1}^{K+1}\lambda_k^{-2} \le \frac{K+1}{eig_{min}\left(\hat{Q}_{x,i}\right)^2} = \frac{K+1}{eig_{max}\left(\hat{Q}_{x,i}\right)^2}CN\left(\hat{Q}_{x,i}\right)^2$, where the λ_k are the eigenvalues of matrix $\hat{Q}_{x,i}$, and $eig_{max}\left(\hat{Q}_{x,i}\right) \ge 1$, we get $\mathbf{1}_i^\chi\|\hat{Q}_{x,i}^{-1}\| \le C\chi_{1,T}$ and:

$$
E\left[\|I_{1211}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\}\right] \le \frac{C\chi_{1,T}^2\chi_{2,T}^4}{nT}\sum_{i,j}\sum_t |\sigma_{ij,t}|\|W_{i,T}\|\|W_{j,T}\|.
$$

Then, from Cauchy-Schwartz inequality and Assumption A.1 c), we get $E\left[\|I_{1211}\|^2|\{\gamma_i\}\right] \le CM\chi_{1,T}^2\chi_{2,T}^4 \sup_i E\left[\|W_{i,T}\|^4|\gamma_i\right]^{1/2}$. From Assumption C.2 we have $\sup_i E\left[\|W_{i,T}\|^4|\gamma_i\right] \le \sup_{\gamma\in[0,1]} E\left[\left\|\frac{1}{T}\sum_t I_t(\gamma)\left(x_t x_t' - Q_x\right)\right\|^4\right] = O(T^{-c})$. It follows $I_{1211} = o_p(1)$. Similarly $I_{1212} = o_p(1)$, and then $I_{121} = o_p(1)$. We can also show that $I_{122} = o_p(1)$, which yields $I_{12} = o_p(1)$. Hence, $I_1 = O_p(1)$.

Consider now I_2. We have:

$$
\begin{aligned}
\frac{1}{n}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_{x,i}^{-1}Y_{i,T}Y_{i,T}'\hat{Q}_{x,i}^{-1} &= \frac{1}{n}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_x^{-1}Y_{i,T}Y_{i,T}'\hat{Q}_x^{-1} + \frac{1}{n}\sum_i \hat{w}_i\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)Y_{i,T}Y_{i,T}'\hat{Q}_x^{-1} \\
&\quad + \frac{1}{n}\sum_i \hat{w}_i\tau_{i,T}^2\hat{Q}_x^{-1}Y_{i,T}Y_{i,T}'\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) \\
&\quad + \frac{1}{n}\sum_i \hat{w}_i\tau_{i,T}^2\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right)Y_{i,T}Y_{i,T}'\left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1}\right) \\
&=: I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
$$

Let us control the four terms. We get $I_{21} = O_p(1)$ by using a decomposition similar to I_{111} and for the leading term $\left\| \frac{1}{n} \sum_i w_i \tau_i^2 \hat{Q}_x^{-1} Y_{i,T} Y_{i,T}^{'} \hat{Q}_x^{-1} \right\| \leq C \left\| \hat{Q}_x^{-1} \right\|^2 \frac{1}{n} \sum_i \|Y_{i,T}\|^2$ and $E \left[\|Y_{i,T}\|^2 | x_{\underline{T}}, I_{\underline{T}}, \{\gamma_i\} \right] \leq C$. Moreover, we get $I_{22} = o_p(1)$ by using a decomposition similar to I_{111} and for the leading term $\left\| \frac{1}{n} \sum_i w_i \tau_i^2 \left(\hat{Q}_{x,i}^{-1} - \hat{Q}_x^{-1} \right) Y_{i,T} Y_{i,T}^{'} \hat{Q}_x^{-1} \right\| \leq C \left\| \hat{Q}_x^{-1} \right\| \chi_{1,T} \frac{1}{n} \sum_i \|Y_{i,T}\|^2$ (see control of term I_{121}). Similarly, we get that $I_{23} = o_p(1)$ and $I_{24} = o_p(1)$. Hence, $I_2 = O_p(1)$.

Finally, we have:

$$
\begin{aligned}
\hat{Q}_b - Q_b &= \left(\frac{1}{n} \sum_i w_i b_i b_i^{'} - Q_b \right) + \frac{1}{n} \sum_i (\hat{w}_i - w_i) b_i b_i^{'} \\
&\quad + \frac{1}{n} \sum_i \hat{w}_i \left(\hat{b}_i - b_i \right) b_i^{'} + \frac{1}{n} \sum_i \hat{w}_i b_i \left(\hat{b}_i - b_i \right)^{'} + \frac{1}{n} \sum_i \hat{w}_i \left(\hat{b}_i - b_i \right) \left(\hat{b}_i - b_i \right)^{'} \\
&= \left(\frac{1}{n} \sum_i w_i b_i b_i^{'} - Q_b \right) + \frac{1}{n} \sum_i (\hat{w}_i - w_i) b_i b_i^{'} + \frac{1}{n\sqrt{T}} \sum_i \hat{w}_i \tau_{i,T} E_2^{'} \hat{Q}_{x,i}^{-1} Y_{i,T} b_i^{'} \\
&\quad + \frac{1}{n\sqrt{T}} \sum_i \hat{w}_i \tau_{i,T} b_i Y_{i,T}^{'} \hat{Q}_{x,i}^{-1} E_2 + \frac{1}{nT} \sum_i \hat{w}_i \tau_{i,T}^2 E_2^{'} \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}^{'} \hat{Q}_{x,i}^{-1} E_2 \\
&=: I_3 + I_4 + I_5 + I_5^{'} + I_6. \qquad (22)
\end{aligned}
$$

From Assumption SC.2, we have $I_3 = o_p(1)$, and $I_4 = o_p(1)$ follows from Lemma 1. Moreover, by similar arguments as for terms I_1 and I_2, we can show that I_5 and I_6 are $o_p(1)$. Then, from Equation (22), we get $\hat{Q}_b - Q_b = o_p(1)$. Thus, from (21) we deduce that $\|\hat{\nu} - \nu\| = O_p \left(\frac{1}{\sqrt{nT}} + \frac{1}{T} \right) = o_p(1)$.

b) Consistency of $\hat{\lambda}$. By Assumption C.1a), we have $\frac{1}{T} \sum_t f_t - E[f_t] = o_p(1)$, and thus $\left\| \hat{\lambda} - \lambda \right\| \leq \|\hat{\nu} - \nu\| + \left\| \frac{1}{T} \sum_t f_t - E[f_t] \right\| = o_p(1)$.

A.2.3 Proof of Proposition 3

a) Asymptotic normality of $\hat{\nu}$. From Equation (21), we have:

$$
\begin{aligned}
\sqrt{nT} \left(\hat{\nu} - \frac{1}{T} \hat{B}_\nu - \nu \right) &= Q_b^{-1} I_1 + \hat{Q}_b^{-1} \frac{1}{\sqrt{nT}} \sum_i \hat{w}_i \tau_{i,T}^2 \left(E_2^{'} \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}^{'} \hat{Q}_{x,i}^{-1} c_\nu - \tau_{i,T}^{-1} E_2^{'} \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} c_{\hat{\nu}} \right) \\
&\quad + o_p(1) \qquad =: Q_b^{-1} I_1 + \hat{Q}_b^{-1} I_7 + o_p(1). \qquad (23)
\end{aligned}
$$

Let us first show that $Q_b^{-1} I_1$ is asymptotically normal. From the proof of Proposition 2 and the properties of the vec operator and Kronecker product, we have:

$$
\begin{aligned}
Q_b^{-1} I_1 &= Q_b^{-1} \left(\frac{1}{\sqrt{n}} \sum_i w_i \tau_i b_i Y_{i,T}' \right) \hat{Q}_x^{-1} c_\nu + o_p(1) = \left(c_\nu' \hat{Q}_x^{-1} \otimes Q_b^{-1} \right) \frac{1}{\sqrt{n}} \sum_i w_i \tau_i vec \left[b_i Y_{i,T}' \right] + o_p(1) \\
&= \left(c_\nu' \hat{Q}_x^{-1} \otimes Q_b^{-1} \right) \frac{1}{\sqrt{n}} \sum_i w_i \tau_i \left(Y_{i,T} \otimes b_i \right) + o_p(1).
\end{aligned}
$$

Then we deduce $Q_b^{-1} I_1 \Rightarrow N(0, \Sigma_\nu)$, by Assumptions A.2a) and C.1a).

Let us now show that $I_7 = o_p(1)$. We have:

$$
\begin{aligned}
I_7 &= \frac{1}{\sqrt{nT}} \sum_i \hat{w}_i \tau_{i,T}^2 E_2' \hat{Q}_{x,i}^{-1} \left(Y_{i,T} Y_{i,T}' - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} c_\nu - \frac{1}{\sqrt{nT}} \sum_i \hat{w}_i \tau_{i,T}^2 E_2' \hat{Q}_{x,i}^{-1} \left(\tau_{i,T}^{-1} \hat{S}_{ii}^0 - S_{ii,T} \right) \hat{Q}_{x,i}^{-1} c_\nu \\
&\quad - \frac{1}{\sqrt{nT}} \sum_i \hat{w}_i \tau_{i,T} E_2' \hat{Q}_{x,i}^{-1} \left(\hat{S}_{ii} - \hat{S}_{ii}^0 \right) \hat{Q}_{x,i}^{-1} c_\nu - \frac{1}{\sqrt{nT}} \sum_i \hat{w}_i \tau_{i,T} E_2' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} \left(c_{\hat{\nu}} - c_\nu \right) \\
&=: I_{71} - I_{72} - I_{73} - I_{74},
\end{aligned}
$$

where $\hat{S}_{ii}^0 = \frac{1}{T_i} \sum_t I_{i,t} \varepsilon_{i,t}^2 x_t x_t'$ and $S_{ii,T} = \frac{1}{T} \sum_t I_{i,t} \sigma_{ii,t} x_t x_t'$. The four terms are bounded in the next Lemma.

Lemma 2 *Under Assumptions C.1a),b), C.3-C.5,* $I_{71} = O_p\left(\frac{1}{\sqrt{T}}\right)$, $I_{72} = O_p\left(\frac{1}{T}\right)$, $I_{73} = O_p\left(\frac{\sqrt{n}}{T\sqrt{T}}\right)$ *and* $I_{74} = O_p\left(\frac{1}{T} + \frac{\sqrt{n}}{T\sqrt{T}}\right)$.

Then, from $n = o\left(T^3\right)$, we get $I_7 = o_p(1)$ and the conclusion follows.

b) Asymptotic normality of $\hat{\lambda}$. We have $\sqrt{T}\left(\hat{\lambda} - \lambda\right) = \frac{1}{\sqrt{T}} \sum_t \left(f_t - E\left[f_t\right]\right) + \sqrt{T}\left(\hat{\nu} - \nu\right)$. By using $\sqrt{T}\left(\hat{\nu} - \nu\right) = O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{T}}\right) = o_p(1)$, the conclusion follows from Assumption A.2b). ■

A.2.4 Proof of Proposition 4

From Proposition 3, we have to show that $\tilde{\Sigma}_\nu - \Sigma_\nu = o_p(1)$. By $\Sigma_\nu = \left(c_\nu' Q_x^{-1} \otimes Q_b^{-1}\right) S_b \left(Q_x^{-1} c_\nu \otimes Q_b^{-1}\right)$ and $\tilde{\Sigma}_\nu = \left(c_{\hat{\nu}}' \hat{Q}_x^{-1} \otimes \hat{Q}_b^{-1}\right) \tilde{S}_b \left(\hat{Q}_x^{-1} c_{\hat{\nu}} \otimes \hat{Q}_b^{-1}\right)$, where $\tilde{S}_b = \frac{1}{n} \sum_{i,j} \hat{w}_i \hat{w}_j \frac{\tau_{i,T} \tau_{j,T}}{\tau_{ij,T}} \tilde{S}_{ij} \otimes \hat{b}_i \hat{b}_j'$, the statement follows if $\tilde{S}_b - S_b = o_p(1)$. The leading term in $\tilde{S}_b - S_b$ is given by $I_8 = \frac{1}{n} \sum_{i,j} w_i w_j \frac{\tau_i \tau_j}{\tau_{ij}} \left(\tilde{S}_{ij} - S_{ij}\right) \otimes b_i b_j'$, while the other ones can be shown to be $o_p(1)$ by arguments similar to the proofs of Propositions 2 and 3. By using that $\tau_i \leq M$, $\tau_{ij} \geq 1$, $w_i \leq M$ and $\|b_i\| \leq M$, $I_8 = o_p(1)$ follows if we show: $\frac{1}{n}\sum_{i,j}\left\|\tilde{S}_{ij} - S_{ij}\right\| = o_p\left(1\right).$ For this purpose, we introduce the following Lemmas 3 and 4 that extend results in Bickel and Levina (2008) from the i.i.d. case to the time series case.

Lemma 3 *Let* $\psi_{nT} = \max_{i,j}\left\|\hat{S}_{ij} - S_{ij}\right\|$, *and* $\Psi_{nT}\left(\delta\right) = \max_{i,j}\mathbb{P}\left[\left\|\hat{S}_{ij} - S_{ij}\right\| \geq \delta\right]$. *Under Assumption A.3,* $\frac{1}{n}\sum_{i,j}\left\|\tilde{S}_{ij} - S_{ij}\right\| = O_p\left(\psi_{nT}n^{\delta}\kappa^{-q} + n^{\delta}\kappa^{1-q} + \psi_{nT}n^2\Psi_{nT}\left(\left(1-v\right)\kappa\right)\right)$, *for any* $v \in (0,1)$.

Lemma 4 *Under Assumptions C.1 and C.3, if* $\kappa = M\sqrt{\frac{\log n}{T^{\eta}}}$ *with* M *large, then* $n^2\Psi_{nT}\left(\left(1-v\right)\kappa\right) = O\left(1\right)$, *for any* $v \in (0,1)$, *and* $\psi_{nT} = O_p\left(\sqrt{\frac{\log n}{T^{\eta}}}\right)$.

From Lemmas 3 and 4, it follows $\frac{1}{n}\sum_{i,j}\left\|\tilde{S}_{ij} - S_{ij}\right\| = O_p\left(\left(\frac{\log n}{T^{\eta}}\right)^{(1-q)/2}n^{\delta}\right) = o_p\left(1\right)$. ■

A.2.5 Proof of Proposition 5

By definition of $\hat{Q}_e$, we get the following result:

Lemma 5 *Under* $\mathcal{H}_0$ *and Assumption A.2a), we have* $\hat{Q}_e = \frac{1}{n}\sum_i \hat{w}_i\left[c_{\hat{\nu}}'\left(\hat{\beta}_i - \beta_i\right)\right]^2 + O_p\left(\frac{1}{nT} + \frac{1}{T^2}\right)$.

From Lemmas 1 and 5, it follows: $\hat{\xi}_{nT} = \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\left\{\left[c_{\hat{\nu}}'\sqrt{T}\left(\hat{\beta}_i - \beta_i\right)\right]^2 - \tau_{i,T}c_{\hat{\nu}}'\hat{Q}_{x,i}^{-1}\hat{S}_{ii}\hat{Q}_{x,i}^{-1}c_{\hat{\nu}}\right\} + o_p\left(1\right)$. By using $\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) = \tau_{i,T}\hat{Q}_{x,i}^{-1}Y_{i,T}$, we get

$$\begin{aligned}
\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2 c_{\hat{\nu}}'\hat{Q}_{x,i}^{-1}\left(Y_{i,T}Y_{i,T}' - \tau_{i,T}^{-1}\hat{S}_{ii}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}} + o_p\left(1\right) \\
&= \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2 c_{\hat{\nu}}'\hat{Q}_{x,i}^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}} - \frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}^2 c_{\hat{\nu}}'\hat{Q}_{x,i}^{-1}\left(\tau_{i,T}^{-1}\hat{S}_{ii} - S_{ii,T}\right)\hat{Q}_{x,i}^{-1}c_{\hat{\nu}} \\
&\quad + o_p\left(1\right) \qquad =: I_{91} + I_{92} + o_p(1).
\end{aligned}$$

We have $I_{91} = \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2 c_{\hat{\nu}}'\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)\hat{Q}_x^{-1}c_{\hat{\nu}} + o_p(1)$ by arguments similar to the proof of Proposition 2 (see control of I_{111}). By using $\tau_{i,T}^{-1}\hat{S}_{ii} - S_{ii,T} = \frac{1}{T}\sum_t I_{i,t}\left(\hat{\varepsilon}_{i,t}^2 - \varepsilon_{i,t}^2\right)x_t x_t' +$

$\frac{1}{T}\sum_t I_{i,t}\left(\varepsilon_{i,t}^2-\sigma_{ii,t}\right)x_t x_t'$ and an argument similar to the proof of Proposition 2 (see control of J_1), we can show that $I_{92} = O_p(\sqrt{n}/T + 1/\sqrt{T})$. By using $n = o(T^2)$, it follows $I_{92} = o_p(1)$. Then, $\hat{\xi}_{nT} = \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2 c_{\hat{\nu}}'\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right)\hat{Q}_x^{-1}c_{\hat{\nu}}+o_p(1)$. By using that $tr\left[A'B\right] = vec\left[A\right]' vec\left[B\right]$, and $vec\left[YY'\right] = (Y\otimes Y)$ for a vector Y, we get

$$
\begin{aligned}
\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2 tr\left[\hat{Q}_x^{-1}c_{\hat{\nu}}c_{\hat{\nu}}'\hat{Q}_x^{-1}\left(Y_{i,T}Y_{i,T}'-S_{ii,T}\right)\right]+o_p(1) \\
&= \left(vec\left[\hat{Q}_x^{-1}c_{\hat{\nu}}c_{\hat{\nu}}'\hat{Q}_x^{-1}\right]\right)'\frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2\left(Y_{i,T}\otimes Y_{i,T}-vec\left[S_{ii,T}\right]\right)+o_p(1).
\end{aligned}
$$

By using Assumption A.4, and by consistency of $\hat{\nu}$ and $\hat{Q}_x$, we get $\hat{\xi}_{nT} \Rightarrow N(0,\Sigma_\xi)$, where $\Sigma_\xi = \left(vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right]\right)'\Omega\left(vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right]\right)$. By using MN Theorem 3 Chapter 2, we have

$$
\begin{aligned}
vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right]'\left(S_{ij}\otimes S_{ij}\right)vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right] &= tr\left[S_{ij}Q_x^{-1}c_\nu c_\nu' Q_x^{-1}S_{ij}Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right] \\
&= \left(c_\nu' Q_x^{-1}S_{ij}Q_x^{-1}c_\nu\right)^2, \qquad (24)
\end{aligned}
$$

and

$$
vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right]'\left(S_{ij}\otimes S_{ij}\right)W_{(K+1)}vec\left[Q_x^{-1}c_\nu c_\nu' Q_x^{-1}\right] = \left(c_\nu' Q_x^{-1}S_{ij}Q_x^{-1}c_\nu\right)^2. \qquad (25)
$$

Then, from the definition of Ω and Equations (24) and (25), we deduce $\Sigma_\xi = 2\operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{i,j}w_i w_j\frac{\tau_i^2\tau_j^2}{\tau_{ij}^2}\left(c_\nu' Q_x^{-1}S_{ij}Q_x^{-1}c_\nu\right)^2$. Finally, $\tilde{\Sigma}_\xi = \Sigma_\xi + o_p(1)$ follows from $\frac{1}{n}\sum_{i,j}\|\tilde{S}_{ij}-S_{ij}\| = o_p(1)$ and $\frac{1}{n}\sum_{i,j}\|\tilde{S}_{ij}-S_{ij}\|^2 = o_p(1)$. ■

A.2.6 Proof of Proposition 6

a) Asymptotic normality of $\hat{\nu}$. By definition of $\hat{\nu}$ and under $\mathcal{H}_1$, we have

$$
\begin{aligned}
\hat{\nu}-\nu_\infty &= \hat{Q}_b^{-1}\frac{1}{n}\sum_i\hat{w}_i\hat{b}_i c_{\nu_\infty}'\hat{\beta}_i = \hat{Q}_b^{-1}\frac{1}{n}\sum_i\hat{w}_i\hat{b}_i c_{\nu_\infty}'\left(\hat{\beta}_i-\beta_i\right)+\hat{Q}_b^{-1}\frac{1}{n}\sum_i\hat{w}_i\hat{b}_i e_i \\
&= \hat{Q}_b^{-1}\frac{1}{n}\sum_i\hat{w}_i\hat{b}_i c_{\nu_\infty}'\left(\hat{\beta}_i-\beta_i\right)+\hat{Q}_b^{-1}\frac{1}{n}\sum_i\hat{w}_i b_i e_i+\hat{Q}_b^{-1}\frac{1}{n}\sum_i\hat{w}_i\left(\hat{b}_i-b_i\right)e_i.
\end{aligned}
$$

Thus we get:

$$\begin{aligned}\sqrt{n}\left(\hat{\nu}-\nu_\infty\right) &= \hat{Q}_b^{-1}\frac{1}{\sqrt{nT}}\sum_i \hat{w}_i\tau_{i,T}\hat{b}_i c'_{\nu_\infty}\hat{Q}_{x,i}^{-1}Y_{i,T}+\hat{Q}_b^{-1}\frac{1}{\sqrt{n}}\sum_i w_i b_i e_i \\ &\quad +\hat{Q}_b^{-1}\frac{1}{\sqrt{n}}\sum_i(\hat{w}_i-w_i)b_i e_i+\hat{Q}_b^{-1}\frac{1}{\sqrt{nT}}\sum_i \hat{w}_i\tau_{i,T}e_i E_2'\hat{Q}_{x,i}^{-1}Y_{i,T} \\ &=: I_{101}+I_{102}+I_{103}+I_{104}.\end{aligned}$$

From Assumption SC.2 and $E_G\left[w_i b_i e_i\right]=0$, we get $\frac{1}{\sqrt{n}}\sum_i w_i b_i e_i \Rightarrow N\left(0,E_G\left[b_i b_i' w_i^2 e_i^2\right]\right)$ by the CLT. Thus $I_{102}\Rightarrow N\left(0,Q_b^{-1}E_G\left[w_i^2 e_i^2 b_i b_i'\right]Q_b^{-1}\right)$. Then the asymptotic distribution of $\hat{\nu}$ follows if terms I_{101}, I_{103} and I_{104} are $o_p(1)$. From similar arguments as in the proof of Proposition 2 (control of term I_1), we have $\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}\hat{b}_i c'_{\nu_\infty}\hat{Q}_{x,i}^{-1}Y_{i,T}=O_p(1)$ and $\frac{1}{\sqrt{n}}\sum_i \hat{w}_i\tau_{i,T}e_i E_2'\hat{Q}_{x,i}^{-1}Y_{i,T}=O_p(1)$. Thus $I_{101}=o_p(1)$ and $I_{104}=o_p(1)$. Moreover, term I_{103} is $o_p(1)$ from Lemma 1.

b) Asymptotic normality of $\hat{\lambda}$. We have $\sqrt{T}\left(\hat{\lambda}-\lambda_\infty\right)=\sqrt{T}\left(\hat{\nu}-\nu_\infty\right)+\frac{1}{\sqrt{T}}\sum_t\left(f_t-E\left[f_t\right]\right)$. By using $\sqrt{T}\left(\hat{\nu}-\nu_\infty\right)=O_p\left(\sqrt{\frac{T}{n}}\right)=o_p(1)$, the conclusion follows.

c) Consistency of the test. By definition of $\hat{Q}_e$, we get the following result:

Lemma 6 *Under $\mathcal{H}_1$ and Assumption A.2a), we have* $\hat{Q}_e=\frac{1}{n}\sum_i \hat{w}_i\left[c'_{\hat{\nu}}\left(\hat{\beta}_i-\beta_i\right)\right]^2+\frac{1}{n}\sum_i \hat{w}_i e_i^2+$ $O_p\left(\frac{1}{\sqrt{nT}}\right)$.

By similar arguments as in the proof of Proposition 4, we get:

$$\begin{aligned}\hat{\xi}_{nT} &= \frac{1}{\sqrt{n}}\sum_i w_i\tau_i^2 c'_{\hat{\nu}}\hat{Q}_x^{-1}\left(Y_{i,T}Y'_{i,T}-S_{ii,T}\right)\hat{Q}_x^{-1}c_{\hat{\nu}}+T\frac{1}{\sqrt{n}}\sum_i w_i e_i^2+O_p\left(\sqrt{T}\right) \\ &= O_p(1)+O\left(T\sqrt{n}E_G\left[w_i\left(a_i-b_i\nu_\infty\right)^2\right]\right)+O_p(T).\end{aligned}$$

Under $\mathcal{H}_1$ we have $E_G\left[w_i\left(a_i-b_i\nu_\infty\right)^2\right]>0$, since $w_i>0$ and $\left(a_i-b_i\nu_\infty\right)^2>0$, P-a.s. ■

Appendix 3: Conditional factor model

A.3.1 Proof of Proposition 7

Proposition 7 is proved along similar lines as Proposition 1. Hence we only highlight the slight differences. We can work at $t = 1$ because of stationarity, and use that $a_1(\gamma)$, $b_1(\gamma)$, for $\gamma \in [0, 1]$, are $\mathcal{F}_0$-measurable. Then the three steps uses again the strong LLN applied conditionally on $\mathcal{F}_0$ and Assumption APR.7 as in the proof of Proposition APR. Equation (8) follows by defining $\nu_t(\omega) = \nu[S^{t-1}(\omega)]$. Uniqueness of vector ν_t is implied by Assumption APR.7.

A.3.2 Derivation of Equations (12) and (13)

From Equation (11) and by using $vec\,[ABC] = \left[C' \otimes A\right] vec\,[B]$ (MN Theorem 2, p. 35), we get $Z'_{t-1}B'_i f_t = vec\left[Z'_{t-1}B'_i f_t\right] = \left[f'_t \otimes Z'_{t-1}\right] vec\left[B'_i\right]$, and $Z'_{i,t-1}C'_i f_t = \left[f'_t \otimes Z'_{i,t-1}\right] vec\left[C'_i\right]$, which gives $Z'_{t-1}B'_i f_t + Z'_{i,t-1}C'_i f_t = x'_{2,i,t}\beta_{2,i}$.

a) By definition of matrix X_t in Section 3.1, we have

$$\begin{aligned} Z'_{t-1}B'_i\left(\Lambda - F\right)Z_{t-1} &= \frac{1}{2}Z'_{t-1}\left[B'_i\left(\Lambda - F\right) + \left(\Lambda - F\right)'B_i\right]Z_{t-1} \\ &= \frac{1}{2}vech\left[X_t\right]'vech\left[B'_i\left(\Lambda - F\right) + \left(\Lambda - F\right)'B_i\right]. \end{aligned}$$

By using the Moore-Penrose inverse of the duplication matrix D_p, we get

$$vech\left[B'_i\left(\Lambda - F\right) + \left(\Lambda - F\right)'B_i\right] = D_p^+\left[vec\left[B'_i\left(\Lambda - F\right)\right] + vec\left[\left(\Lambda - F\right)'B_i\right]\right].$$

Finally, by the properties of the vec operator and the commutation matrix $W_{p,K}$, we obtain

$$\frac{1}{2}D_p^+\left[vec\left[B'_i\left(\Lambda - F\right)\right] + vec\left[\left(\Lambda - F\right)'B_i\right]\right] = \frac{1}{2}D_p^+\left[\left(\Lambda - F\right)' \otimes I_p + I_p \otimes \left(\Lambda - F\right)'W_{p,K}\right]vec\left[B'_i\right].$$

b) By definition of matrix $X_{i,t}$ in Section 3.1, we have

$$Z'_{i,t-1}C'_i\left(\Lambda - F\right)Z_{t-1} = vec\left[Z_{t-1}Z'_{i,t-1}\right]'vec\left[C'_i\left(\Lambda - F\right)\right] = vec\left[X_{i,t}\right]'\left[\left(\Lambda - F\right)' \otimes I_q\right]vec\left[C'_i\right].$$

By combining a) and b), we deduce $Z'_{t-1}B'_i\left(\Lambda - F\right)Z_{t-1} + Z'_{i,t-1}C'_i\left(\Lambda - F\right)Z_{t-1} = x'_{1,i,t}\beta_{1,i}$ and $\beta_{1,i} = \Psi\beta_{2,i}$.

A.3.3 Derivation of Equation (14)

a) From the properties of the vec operator, we get

$$vec\left[B_i'\left(\Lambda - F\right)\right] + vec\left[\left(\Lambda - F\right)' B_i\right] = \left(I_p \otimes B_i'\right) vec\left[\Lambda - F\right] + \left(B_i' \otimes I_p\right) vec\left[\Lambda' - F'\right].$$

Since $vec\left[\Lambda - F\right] = W_{p,K} vec\left[\Lambda' - F'\right]$, we can factorize $\nu = vec\left[\Lambda' - F'\right]$ to obtain

$$\frac{1}{2} D_p^+ \left[vec\left[B_i'\left(\Lambda - F\right)\right] + vec\left[\left(\Lambda - F\right)' B_i\right]\right] = \frac{1}{2} D_p^+ \left[\left(I_p \otimes B_i'\right) W_{p,K} + B_i' \otimes I_p\right] \nu.$$

By properties of commutation and duplication matrices (MN p. 54-58), we have $\left(I_p \otimes B_i'\right) W_{p,K} = W_p \left(B_i' \otimes I_p\right)$ and $D_p^+ W_p = D_p^+$, then $\frac{1}{2} D_p^+ \left[\left(I_p \otimes B_i'\right) W_{p,K} + B_i' \otimes I_p\right] = D_p^+ \left(B_i' \otimes I_p\right)$.

b) From the properties of the vec operator, we get

$$vec\left[C_i'\left(\Lambda - F\right)\right] = \left(I_p \otimes C_i'\right) vec\left[\Lambda - F\right] = \left(I_p \otimes C_i'\right) W_{p,K} vec\left[\Lambda' - F'\right] = W_{p,q}\left(C_i' \otimes I_p\right) \nu.$$

A.3.4 Derivation of Equation (15)

a) By MN Theorem 2 p. 35 and Exercise 1 p. 56, and by writing $I_{pK} = I_K \otimes I_p$, we obtain

$$\begin{aligned} vec\left[D_p^+\left(B_i' \otimes I_p\right)\right] &= \left(I_{pK} \otimes D_p^+\right) vec\left[B_i' \otimes I_p\right] \\ &= \left(I_{pK} \otimes D_p^+\right)\left\{I_K \otimes \left[\left(W_p \otimes I_p\right)\left(I_p \otimes vec\left[I_p\right]\right)\right]\right\} vec\left[B_i'\right] \\ &= \left\{I_K \otimes \left[\left(I_p \otimes D_p^+\right)\left(W_p \otimes I_p\right)\left(I_p \otimes vec\left[I_p\right]\right)\right]\right\} vec\left[B_i'\right]. \end{aligned}$$

Moreover, $vec\left[\left\{D_p^+\left(B_i' \otimes I_p\right)\right\}'\right] = W_{p(p+1)/2,pK} vec\left[D_p^+\left(B_i' \otimes I_p\right)\right]$.

b) Similarly, $vec\left[W_{p,q}\left(C_i' \otimes I_p\right)\right] = \left\{I_K \otimes \left[\left(I_p \otimes W_{p,q}\right)\left(W_{p,q} \otimes I_p\right)\left(I_q \otimes vec\left[I_p\right]\right)\right]\right\} vec\left[C_i'\right]$ and $vec\left[\left\{W_{p,q}\left(C_i' \otimes I_p\right)\right\}'\right] = W_{pq,pK} vec\left[W_{p,q}\left(C_i' \otimes I_p\right)\right]$.

By combining a) and b) and using $vec\left[\beta_{3,i}'\right] = \left(vec\left[\left\{D_p^+\left(B_i' \otimes I_p\right)\right\}'\right]', vec\left[\left\{W_{p,q}\left(C_i' \otimes I_p\right)\right\}'\right]'\right)'$ the conclusion follows.

A.3.5 Proof of Proposition 8

a) Consistency of $\hat{\nu}$. By definition of $\hat{\nu}$ we have: $\hat{\nu} - \nu = \hat{Q}_{\beta_3}^{-1} \frac{1}{n} \sum_i \hat{\beta}_{3,i}' \hat{w}_i \left(\hat{\beta}_{1,i} - \hat{\beta}_{3,i}\nu\right)$. From Equation (15) and MN Theorem 2 p. 35, we get $\hat{\beta}_{3,i}\nu = vec[\nu'\hat{\beta}_{3,i}'] = \left(I_{d_1} \otimes \nu'\right) vec[\hat{\beta}_{3,i}'] = \left(I_{d_1} \otimes \nu'\right) J_a \hat{\beta}_{2,i}$.

Moreover, by using matrices E_1 and E_2, we obtain $\left(\hat{\beta}_{1,i}-\hat{\beta}_{3,i}\nu\right)=\left[E_1'-\left(I_{d_1}\otimes\nu'\right)J_aE_2'\right]\hat{\beta}_i=C_\nu'\hat{\beta}_i=C_\nu'\left(\hat{\beta}_i-\beta_i\right)$, from Equation (14). It follows that

$$\hat{\nu}-\nu \quad = \quad \hat{Q}_{\beta_3}^{-1}\frac{1}{n}\sum_i\hat{\beta}_{3,i}'\hat{w}_iC_\nu'\left(\hat{\beta}_i-\beta_i\right). \tag{26}$$

By comparing with Equation (20) and using the same arguments as in the proof of Proposition 1 applied to β_3' instead of b, the result follows.

b) Consistency of Λ'. By definition of ν, we deduce $\left\|vec\left[\hat{\Lambda}'-\Lambda'\right]\right\|\leq\|\hat{\nu}-\nu\|+\left\|vec\left[\hat{F}'-F'\right]\right\|$. By part a), $\|\hat{\nu}-\nu\|=o_p\left(1\right)$. By LLN and Assumptions C.1a),b) and C.6, we have $\frac{1}{T}\sum_tZ_{t-1}Z_{t-1}'=O_p\left(1\right)$ and $\frac{1}{T}\sum_tu_tZ_{t-1}'=o_p\left(1\right)$. Then, by Slustky theorem, we conclude that $\left\|vec\left[\hat{F}'-F'\right]\right\|=o_p\left(1\right)$. The result follows.

A.3.6 Proof of Proposition 9

a) Asymptotic normality of $\hat{\nu}$. From Equation (26) and by using $\sqrt{T}\left(\hat{\beta}_i-\beta_i\right)=\tau_{i,T}\hat{Q}_{x,i}^{-1}Y_{i,T}$, we get

$$\begin{aligned}\sqrt{nT}\left(\hat{\nu}-\nu\right) &= \hat{Q}_{\beta_3}^{-1}\frac{1}{\sqrt{n}}\sum_i\tau_{i,T}\hat{\beta}_{3,i}'\hat{w}_iC_\nu'\hat{Q}_{x,i}^{-1}Y_{i,T}\\ &= \hat{Q}_{\beta_3}^{-1}\frac{1}{\sqrt{n}}\sum_i\tau_{i,T}\beta_{3,i}'\hat{w}_iC_\nu'Q_{x,i}^{-1}Y_{i,T}+\hat{Q}_{\beta_3}^{-1}\frac{1}{\sqrt{n}}\sum_i\tau_{i,T}\beta_{3,i}'\hat{w}_iC_\nu'\left(\hat{Q}_{x,i}^{-1}-Q_{x,i}^{-1}\right)Y_{i,T}\\ &\quad+\hat{Q}_{\beta_3}^{-1}\frac{1}{\sqrt{n}}\sum_i\tau_{i,T}\left(\hat{\beta}_{3,i}-\beta_{3,i}\right)'\hat{w}_iC_\nu'\hat{Q}_{x,i}^{-1}Y_{i,T}\quad=:I_{71}+I_{72}+I_{73}.\end{aligned}$$

By MN Theorem 2 p. 35, we have $I_{71}=\hat{Q}_{\beta_3}^{-1}\left(\frac{1}{\sqrt{n}}\sum_i\tau_{i,T}\left[(Y_{i,T}'Q_{x,i}^{-1})\otimes(\beta_{3,i}'\hat{w}_i)\right]\right)vec\left[C_\nu'\right]$. As in the proof of Propositions 2 and 3, we have $I_{71}=\hat{Q}_{\beta_3}^{-1}\left(\frac{1}{\sqrt{n}}\sum_i\tau_i\left[(Y_{i,T}'Q_{x,i}^{-1})\otimes(\beta_{3,i}'w_i)\right]\right)vec\left[C_\nu'\right]$ $+o_p(1)=:I_{711}+o_p(1)$. We can rewrite $I_{711}=\left(vec\left[C_\nu'\right]'\otimes\hat{Q}_{\beta_3}^{-1}\right)\frac{1}{\sqrt{n}}\sum_i\tau_ivec\left[(Y_{i,T}'Q_{x,i}^{-1})\otimes(\beta_{3,i}'w_i)\right]$.

Moreover, by using $vec\left[(Y_{i,T}'Q_{x,i}^{-1})\otimes(\beta_{3,i}'w_i)\right]=(Q_{x,i}^{-1}Y_{i,T})\otimes vec\left[\beta_{3,i}'w_i\right]$ (see MN Theorem 10 p. 55), we get $I_{711}=\left(vec\left[C_\nu'\right]'\otimes\hat{Q}_{\beta_3}^{-1}\right)\frac{1}{\sqrt{n}}\sum_i\tau_i\left[(Q_{x,i}^{-1}Y_{i,T})\otimes v_3\right]$. Then $I_{711}\Rightarrow N\left(0,\Sigma_\nu\right)$ follows from Assumption B.2 a).

Let us consider I_{72}. By similar arguments as in the proof of Proposition 3, $I_{72} = o_p(1)$.

Let us consider I_{73}. We introduce the following lemma:

Lemma 7 *Let A be a $m \times n$ matrix and b be a $n \times 1$ vector. Then,* $Ab = \left(vec\left[I_n\right]' \otimes I_m\right) vec\left[vec\left[A\right] b'\right]$.

By Lemma 7, Equation (15) and $\sqrt{T} vec\left[\left(\hat{\beta}_{3,i} - \beta_{3,i}\right)'\right] = \tau_{i,T} J_a E_2' \hat{Q}_{x,i}^{-1} Y_{i,T}$, we have

$$\begin{aligned} I_{73} &= \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T}^2 \left(vec\left[I_{d_1}\right]' \otimes I_{Kp}\right) vec\left[J_a E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_\nu \hat{w}_i\right] \\ &= \hat{Q}_{\beta_3}^{-1} \frac{1}{\sqrt{nT}} \sum_i \tau_{i,T}^2 J_b vec\left[E_2' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_\nu \hat{w}_i\right] \qquad =: \sqrt{\frac{n}{T}} \hat{B}_\nu + I_{74}, \end{aligned}$$

where $I_{74} = o_p(1)$ by similar arguments as in the proof of Proposition 3.

b) Asymptotic normality of $vec\left(\hat{\Lambda}'\right)$**.** We have $\sqrt{T} vec\left[\hat{\Lambda}' - \Lambda'\right] = \sqrt{T} vec\left[\hat{F}' - F'\right] + \sqrt{T}\left(\hat{\nu} - \nu\right)$. By using $\sqrt{T} vec\left[\hat{F}' - F'\right] = \left[I_K \otimes \left(\frac{1}{T} \sum_t Z_{t-1} Z_{t-1}'\right)^{-1}\right] \frac{1}{\sqrt{T}} \sum_t u_t \otimes Z_{t-1}$ and $\sqrt{T}\left(\hat{\nu} - \nu\right) = O_p\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{T}}\right) = o_p(1)$, the conclusion follows from Assumption B.2b). ■

A.3.7 Proof of Proposition 10

By similar arguments as in the proof of Proposition 5, we have:

$$\begin{aligned} \hat{Q}_e &= \frac{1}{n} \sum_i \left(\hat{\beta}_i - \beta_i\right)' C_{\hat{\nu}} \hat{w}_i C_{\hat{\nu}}' \left(\hat{\beta}_i - \beta_i\right) + O_p\left(\frac{1}{nT} + \frac{1}{T^2}\right) \\ &= \frac{1}{nT} \sum_i \tau_{i,T}^2 tr\left[C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} Y_{i,T} Y_{i,T}' \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i\right] + O_p\left(\frac{1}{nT} + \frac{1}{T^2}\right). \end{aligned}$$

By using that $\tau_{i,T} tr\left[C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1} \hat{S}_{ii} \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i\right] = \mathbf{1}_i^\chi d_1$ and Lemma 1 in the conditional case, we get:

$$\begin{aligned} \hat{\xi}_{nT} &= \frac{1}{\sqrt{n}} \sum_i \tau_{i,T}^2 tr\left[C_{\hat{\nu}}' \hat{Q}_{x,i}^{-1}\left(Y_{i,T} Y_{i,T}' - \tau_{i,T}^{-1} \hat{S}_{ii}\right) \hat{Q}_{x,i}^{-1} C_{\hat{\nu}} \hat{w}_i\right] + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_i \tau_i^2 tr\left[C_{\hat{\nu}}' Q_{x,i}^{-1}\left(Y_{i,T} Y_{i,T}' - S_{ii,T}\right) Q_{x,i}^{-1} C_{\hat{\nu}} w_i\right] + o_p(1). \end{aligned}$$

Now, by using $tr(ABCD) = vec(D')'(C' \otimes A)vec(B)$ (MN Theorem 3, p. 31) and $vec(ABC) = (C' \otimes A)vec(B)$ for conformable matrices, we have:

$$\begin{aligned} & tr\left[C_{\hat{\nu}}' Q_{x,i}^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)Q_{x,i}^{-1}C_{\hat{\nu}}w_i\right] \\ = \; & vec[w_i]'\left(C_{\hat{\nu}}' \otimes C_{\hat{\nu}}'\right) vec\left[Q_{x,i}^{-1}\left(Y_{i,T}Y_{i,T}' - S_{ii,T}\right)Q_{x,i}^{-1}\right] \\ = \; & vec[w_i]'\left(C_{\hat{\nu}}' \otimes C_{\hat{\nu}}'\right)\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right) vec\left[Y_{i,T}Y_{i,T}' - S_{ii,T}\right] \\ = \; & vec[w_i]'\left(C_{\hat{\nu}}' \otimes C_{\hat{\nu}}'\right)\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)\left(Y_{i,T} \otimes Y_{i,T} - vec[S_{ii,T}]\right) \\ = \; & vec\left[C_{\hat{\nu}}' \otimes C_{\hat{\nu}}'\right]'\left\{\left[\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)\left(Y_{i,T} \otimes Y_{i,T} - vec[S_{ii,T}]\right)\right] \otimes vec[w_i]\right\}. \end{aligned}$$

Thus, we get $\xi_{nT} = vec\left[C_{\hat{\nu}}' \otimes C_{\hat{\nu}}'\right]' \frac{1}{\sqrt{n}} \sum_i \tau_i^2 \left[\left(Q_{x,i}^{-1} \otimes Q_{x,i}^{-1}\right)\left(Y_{i,T} \otimes Y_{i,T} - vec\left[S_{ii,T}\right]\right)\right] \otimes vec[w_i]$. From Assumption B.3, we get $\hat{\xi}_{nT} \Rightarrow N(0, \Sigma_\xi)$, where $\Sigma_\xi = vec\left[C_\nu' \otimes C_\nu'\right]' \Omega vec\left[C_\nu' \otimes C_\nu'\right]$. Now, by using that $tr(ABCD) = vec(D)'(A \otimes C')vec(B')$ (see Theorem 3, p. 31, in MN) we have:

$$\begin{aligned} & vec\left[C_\nu' \otimes C_\nu'\right]'\left[(S_{Q,ij} \otimes S_{Q,ij}) \otimes vec[w_i]vec[w_j]'\right] vec\left[C_\nu' \otimes C_\nu'\right] \\ = \; & tr\left[(S_{Q,ij} \otimes S_{Q,ij})(C_\nu \otimes C_\nu) vec[w_j]vec[w_i]'\left(C_\nu' \otimes C_\nu'\right)\right] \\ = \; & vec[w_i]'\left[\left(C_\nu' S_{Q,ij} C_\nu\right) \otimes \left(C_\nu' S_{Q,ij} C_\nu\right)\right] vec[w_j] \\ = \; & tr\left[\left(C_\nu' S_{Q,ij} C_\nu\right) w_j \left(C_\nu' S_{Q,ji} C_\nu\right) w_i\right] \\ = \; & tr\left[\left(C_\nu' Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1} C_\nu\right) w_j \left(C_\nu' Q_{x,j}^{-1} S_{ji} Q_{x,i}^{-1} C_\nu\right) w_i\right], \end{aligned}$$

and similarly $vec\left[C_\nu' \otimes C_\nu'\right]'\left[(S_{Q,ij} \otimes S_{Q,ij})W_d \otimes vec[w_i]vec[w_j]'\right] vec\left[C_\nu' \otimes C_\nu'\right]$ $= tr\left[\left(C_\nu' Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1} C_\nu\right) w_j \left(C_\nu' Q_{x,j}^{-1} S_{ji} Q_{x,i}^{-1} C_\nu\right) w_i\right]$. Thus, we get the asymptotic variance matrix $\Sigma_\xi = 2 \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i,j} \frac{\tau_i^2 \tau_j^2}{\tau_{ij}^2} tr\left[\left(C_\nu' Q_{x,i}^{-1} S_{ij} Q_{x,j}^{-1} C_\nu\right) w_j \left(C_\nu' Q_{x,j}^{-1} S_{ji} Q_{x,i}^{-1} C_\nu\right) w_i\right]\right]$. From $\tilde{\Sigma}_\xi = \Sigma_\xi + o_p(1)$, the conclusion follows. ■

A.3.8 Proof of Equation (17)

We have:

$$\hat{b}_{i,t}' \hat{\lambda}_t = tr\left[Z_{t-1} Z_{t-1}' \hat{B}_i' \hat{\Lambda}\right] + tr\left[Z_{t-1} Z_{i,t-1}' \hat{C}_i' \hat{\Lambda}\right] = \left(Z_{t-1}' \otimes Z_{t-1}'\right) vec\left[\hat{B}_i' \hat{\Lambda}\right] + \left(Z_{t-1}' \otimes Z_{i,t-1}'\right) vec\left[\hat{C}_i' \hat{\Lambda}\right].$$

Thus, we get:

$$
\begin{aligned}
& \sqrt{T}\left(\widehat{CE}_{i,t} - CE_{i,t}\right) \\
= \; & \left(Z'_{t-1} \otimes Z'_{t-1}\right) \sqrt{T}\left(vec\left[\hat{B}'_i \hat{\Lambda}\right] - vec\left[B'_i \Lambda\right]\right) + \left(Z'_{t-1} \otimes Z'_{i,t-1}\right) \sqrt{T}\left(vec\left[\hat{C}'_i \hat{\Lambda}\right] - vec\left[C'_i \Lambda\right]\right) \\
= \; & \left(Z'_{t-1} \otimes Z'_{t-1}\right)\left[\left(\hat{\Lambda}' \otimes I_p\right) \sqrt{T} vec\left[\hat{B}'_i - B'_i\right] + \left(I_p \otimes B'_i\right) \sqrt{T} vec\left[\hat{\Lambda} - \Lambda\right]\right] \\
& + \left(Z'_{t-1} \otimes Z'_{i,t-1}\right)\left[\left(\hat{\Lambda}' \otimes I_q\right) \sqrt{T} vec\left[\hat{C}'_i - C'_i\right] + \left(I_p \otimes C'_i\right) \sqrt{T} vec\left[\hat{\Lambda} - \Lambda\right]\right].
\end{aligned}
$$

By using that $\hat{\Lambda} = \Lambda + o_p(1)$ and $vec\left[\hat{\Lambda} - \Lambda\right] = W_{p,K} vec\left[\hat{\Lambda}' - \Lambda'\right]$, Equation (17) follows. ■

Appendix 4: Check of assumptions under block dependence

In this appendix we verify that the eigenvalue condition in APR.4 (i) and the cross-sectional dependence and asymptotic normality conditions in Assumptions A.1-A.4 are satisfied under a block-dependence structure in a serially i.i.d. framework. Let us assume that:

BD.1 The errors $\varepsilon_t(\gamma)$ are i.i.d. over time with $E[\varepsilon_t(\gamma)] = 0$, for all $\gamma \in [0, 1]$. For any n, there exists a partition of the interval $[0, 1]$ into $J_n \leq n$ subintervals $I_1, ..., I_{J_n}$, such that $\varepsilon_t(\gamma)$ and $\varepsilon_t(\gamma')$ are independent if γ and γ' belong to different subintervals, and $J_n \to \infty$ as $n \to \infty$.

BD.2 The blocks are such that $\max_{m=1,...,J_m} B_m = o(1)$, $n \sum_{m=1}^{J_n} |B_m|^2 = O(1)$, $n^{3/2} \sum_{m=1}^{J_n} |B_m|^3 = o(1)$, where $B_m = \int_{I_m} dG(\gamma)$.

BD.3 The factors (f_t) are i.i.d. over time and independent of the errors $(\varepsilon_t(\gamma))$, $\gamma \in [0, 1]$.

BD.4 There exists a constant M such that $\|f_t\| \leq M$, P-a.s.. Moreover, $\sup_{\gamma\in[0,1]} E[|\varepsilon_t(\gamma)|^6] < \infty$, $\sup_{\gamma\in[0,1]} \|\beta(\gamma)\| < \infty$ and $\inf_{\gamma\in[0,1]} E[I_t(\gamma)] > 0$.

The block-dependence structure as in Assumption BD.1 is satisfied for instance when there are unobserved industry-specific factors independent among industries and over time, as in Ang, Liu, Schwartz (2010). In empirical applications, blocks can match industrial sectors. Then, the number J_n of blocks amounts to a couple of dozens, and the number of assets n amounts to a couple of thousands. There are approximately nB_m assets in block m, when n is large. In the asymptotic analysis, Assumption BD.2 on block sizes and block number requires that the largest block size shrinks with n and that there are not too many large blocks, i.e., the partition in independent blocks is sufficiently fine grained asymptotically. Within blocks, covariances do not need to vanish asymptotically.

Lemma 8 *Let Assumptions BD.1-4 on block dependence and Assumptions SC.1-SC.2 on random sampling hold. Then, Assumption APR.4 (i) is satisfied, and Assumptions A.1, A.2 (with* $\Gamma_1 = \mathbb{R}^+$*), A.3 (with any* $q \in (0, 1)$ *and* $\delta = 1/2$*) and A.4 (with* $\Gamma_2 = \mathbb{R}^+$*) are satisfied.*

In Lemma 8, we have $\Gamma_1 = \Gamma_2 = \mathbb{R}^+$, which means that there is no condition on the relative expansion rates of n and T. The proof of Lemma 8 uses results in Stout (1974) and Bosq (1998).

Instead of having a block structure, we can also assume that the covariance matrix is full, but with off-diagonal elements vanishing asymptotically. In that setting, we can carry out similar checks.