ECG752 - Time Series Econometrics - Spring 2009
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Problem Set 3

Question 1

Consider a GARCH(1,1) model with Gaussian innovations:

$$\begin{aligned} y_t &= \sigma_t u_t & (1) \\ \sigma_t^2 &= \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 & (2) \end{aligned}$$

wit u_t i.i.d. $N(0,1)$. Let us assume that the parameters of the model are such that the process is fourth order stationary:

$$\begin{aligned} E[y_t^i] &= m_i < +\infty \quad \forall t \ , i = 1,2,3,4 \\ E[y_t y_{t-i} y_{t-j} y_{t-k}] &= f(i,j,k) \quad \forall t \end{aligned}$$

Question 1.1 Compute the fourth moment of y_t, *i.e.* $E[y_t^4]$.
Question 1.2 Study the set of values for α and β for which $E[y_t^4]$ is finite.
Question 1.3 Does the GARCH(1,1) estimated on the daily NASDAQ index in Computer Exercise 3 has a finite fourth moment?

Question 2

Consider an ARMA(p, q) model

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + u_t + \theta_1 u_{t-1} + \cdots + \theta_q u_{t-q} \quad (3)$$

where the roots of the AR polynomial, $P(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$, and the MA polynomial, $Q(L) = 1 + \sum_{i=1}^{q} \theta_i L^i$, are outside the unit circle. The variance of the white noise u_t is σ_u^2.

Write enough moment conditions so that this model could be estimated by GMM.

Question 3

You have T independent draws (observations) from a Chi-square distribution and you are interested in estimating the number of degrees of freedom n of this distribution. Knowing that if $X \sim \chi^2_{(n)}$, then

$$\begin{aligned} E[X] &= n \\ E[X^2] &= n(n+2) \\ E[X^3] &= n(n+2)(n+4) \\ E[X^4] &= n(n+2)(n+4)(n+6), \end{aligned}$$

you decide to estimate to estimate n by GMM using your knowledge about the first two moments ($E[X]$ and $E[X^2]$).

Question 3.1 Write down the two moment conditions using the notation we used in class, *i.e.* $h(v_t, \theta)$.

Question 3.2 Compute the optimal weighting matrix that you would employ in the second step of the two-step GMM estimation. What is the impact of n on this weighting matrix? Compare the matrices when $n = 1$ and $n = 100$.

Question 3.3 Another approach you could use to estimate n would be Maximum Likelihood. Write the log-likelihood for this case and compute the First Order Condition (FOC). Do you think there should be a relationship between this FOC and the above GMM estimation?