ECG752 - Time Series Econometrics - Spring 2009
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Problem Set 2

Question 1.

Consider the following matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Compute:

1. $vec(B)$
2. $vec(B')$
3. $vec(B')'$
4. $A \otimes B$
5. $B \otimes A$
6. A^2
7. $A \otimes A$

Question 2.

Take the handout I distributed in class on the estimation by maximum likelihood of a VAR(p) conditional on initial values $y_p, y_{p-1}, \ldots, y_1$. Explain why $vec(\hat{B}) = [I_N \otimes (ZZ')^{-1}Z]vec(Y')$ is "OLS equation by equation". To do so, I suggest studying successively

- $(ZZ')^{-1}Z$
- $I_N \otimes (ZZ')^{-1}Z$
- $[I_N \otimes (ZZ')^{-1}Z]vec(Y')$

Question 3.

Consider the following bivariate VAR(1) process $y_t = \Phi y_{t-1} + a_t$ where

$$\Phi = \begin{bmatrix} 0.8 & -0.1 \\ -0.2 & 0.6 \end{bmatrix}$$

1. Show that this model is stationary.

2. Compute the VMA representation.

Question 4.

Consider a VAR(3) model,

$$y_t = C + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \Phi_2 y_{t-2} + a_t$$

Let us assume that the Φ_i matrices are such that the process is stationary and that this solution is only a function of past and current ϵ_t's. Derive the Ψ_i's, *i.e.* the VMA representation of this VAR.

Question 5.

Let the 2×1 vector $y_t = [y_{1,t}, y_{2,t}]'$ be generated by the VAR(1) process

$$y_t = \Phi y_{t-1} + u_t \tag{1}$$

where

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

and $u_t \sim i.i.d.(0_{2\times 1}, \Sigma_u)$.

1. Show that $y_{i,t}$ has the following univariate representation

$$y_{i,t} = (\phi_{11} + \phi_{22})y_{i,t-1} - (\phi_{11}\phi_{22} - \phi_{12}\phi_{21})y_{i,t-2} + e_{i,t} \tag{2}$$

   where

$$\begin{aligned} e_{1,t} &= u_{1,t} - \phi_{22}u_{1,t-1} + \phi_{12}u_{2,t-1} \\ e_{2,t} &= u_{2,t} - \phi_{11}u_{2,t-1} + \phi_{21}u_{1,t-1} \end{aligned}$$

2. Show that both $e_{1,t}$ and $e_{2,t}$ are stationary and, in general, possess the autocorrelation of a MA(1) process.

3. Given that $y_{i,t}$ has an ARMA(p,q) representation, suggest values for p and q.

4. Show that the condition for stationarity of the VAR(1) is equivalent to the stationarity conditions for the univariate models for $y_{1,t}$ and $y_{2,t}$.

Question 6.

Consider the model

$$y_t = \alpha_0 + \delta_0 t + u_t, \quad t = 1, 2, \ldots, T.$$

Assume that $u_t \sim i.i.d.(0, \sigma_0^2)$. Define $D_T = diag(T^{1/2}, T^{3/2})$, X to be the $T \times 2$ matrix with t^{th} row $(1, t)$ and u to be the $T \times 1$ vector with t^{th} element u_t. Assume that $D_T^{-1} X' u \xrightarrow{d} N(0, \sigma_0^2 Q)$ where $\lim_{T\to\infty} D_T^{-1} X' X D_T^{-1} = Q$. Let $\hat{\beta}_T = (\hat{\alpha}_T, \hat{\delta}_T)'$ be the OLS estimator of $\beta_0 = (\alpha_0, \delta_0)'$ and σ_T^2 be the OLS estimator of σ_0^2.

1. Show that $\sigma_T^2 \xrightarrow{p} \sigma_0^2$.
2. Let $\tau_\alpha = (\hat{\alpha}_T - \alpha_0)/\sqrt{\hat{\sigma}_T^2 \hat{m}_{1,1}}$ and $\tau_\delta = (\hat{\delta}_T - \delta_0)/\sqrt{\hat{\sigma}_T^2 \hat{m}_{2,2}}$ where $\hat{m}_{i,i}$ is the element (i, i) of $(X'X)^{-1}$. Show that $\tau_\alpha \xrightarrow{d} N(0, 1)$ and $\tau_\delta \xrightarrow{d} N(0, 1)$.
3. Suppose we want to test $H_0 : a_1\alpha_0 + a_2\delta_0 = r$. Consider the statistic

   $$\tau_1 = \frac{a_1\hat{\alpha}_T + a_2\hat{\delta}_T - r}{\sqrt{\hat{\sigma}_T^2 a'(X'X)^{-1} a}}$$

   where $a = (a_1, a_2)'$. Show that under H_0, $T^{1/2}(a_1\hat{\alpha}_T - a_2\hat{\delta}_T - r) - T^{1/2}(\hat{\alpha}_T - \alpha_0) = o_p(1)$ and $T\hat{\sigma}_T^2 a'(X'X)^{-1}a - a_1^2\sigma_0^2 m_{1,1} = o_p(1)$ where $m_{1,1}$ is the element $(1, 1)$ element of Q^{-1}, and hence that $\tau_1 \xrightarrow{d} N(0, 1)$.
4. Suppose we want to test $H_0 : \alpha_0 = \bar{\alpha}, \delta_0 = \bar{\delta}$. Consider the statistic

   $$F_2 = [\hat{\beta}_T - \bar{\beta}]' \left[\hat{\sigma}_T^2 (X'X)^{-1}\right]^{-1} [\hat{\beta}_T - \bar{\beta}]$$

   where $\bar{\beta} = (\bar{\alpha}, \bar{\delta})'$. Show that under H_0, $F_2 \xrightarrow{d} \chi_2^2$.