ECG752 - Time Series Econometrics - Spring 2009
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Problem Set 1

Question 1. Consider the following process,

$$X_t = \sum_{j=1}^{m} [A_j \cos(v_j t) + B_j \sin(v_j t)], \quad t \in \mathbb{Z}$$

where $v_1, \ldots, v_m$ are distinct constants in the interval $[0, 2\pi[$ and $A_j, B_j, j = 1, \ldots, m$ are weakly stationary random variables such that

$$\begin{aligned} E[A_j] = E[B_j] &= 0, \quad E[A_j^2] = E[B_j^2] = \sigma_j^2, j = 1, 2, \ldots, m, \\ E[A_j A_k] &= E[B_j B_k] = 0, \text{ for } j \neq k \\ E[A_j B_k] &= 0 \ \forall j, k \end{aligned}$$

1. Show that this process is weakly stationary.
2. For the case $m = 1$, show that this process is deterministic (you can know for sure future values of the process using past and current values).

Hint: Remember the following trigonometric identities:

$$\begin{aligned} 1 &= \cos(x)^2 + \sin(x)^2 \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\ \cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \end{aligned}$$

Question 2. Consider the following ARMA(2,2) process, where u_t is a white noise i.i.d. $N(0,1)$:

$$y_t = 10 + 0.7y_{t-1} - 0.2y_{t-2} + u_t + 0.5u_{t-1} + 0.25u_{t-2}.$$

1. Is the process stationary? Why?
2. Is the process invertible? Why?
3. Compute $E[y_t]$.
4. Compute the autocovariances $\gamma(h)$ for $h = 0, 1, \ldots, 6$.
5. Compute the coefficients of u_t, u_{t-1}, u_{t-6} in the MA(∞) representation of y_t.

Question 3. Check that the following two AR(1) processes

$$\begin{aligned} y_t &= \phi y_{t-1} + u_t, \\ x_t &= \frac{1}{\phi} x_{t-1} + v_t, \end{aligned}$$

with $|\phi| < 1$, have the same autocorrelation functions.

Question 4. Let u_t be a white noise process with $Var[u_t] = \sigma_u^2$. Also let y_t be a stationary process such that $\lim_{T\to\infty} Var[T^{-1/2}\sum_{t=1}^{T} y_t] = \sum_{i=-\infty}^{\infty} \gamma_i$ where $\gamma_i = Cov(y_t, y_{t-i})$. For brevity, denote $S = \lim_{T\to\infty} Var[T^{-1/2}\sum_{t=1}^{T} y_t]$.

1. Let $y_t = Q(L)u_t$, where $Q(L) = 1 + \theta L$. Show that $S = \sigma_u^2[Q(1)]^2$.
2. Let $P(L)y_t = u_t$, where $P(L) = 1 - \phi L$ and $|\theta| < 1$. Show that $S = \sigma_u^2[P(1)]^{-2}$.
3. Let $P(L)y_t = Q(L)u_t$, where $P(L) = 1 - \phi L$, $Q(L) = 1 + \theta L$, $|\phi| < 1$ and $\theta \neq -\phi$. Show that $S = \sigma_u^2[P(1)]^2/[Q(1)]^{-2}$.

Question 5. Take $\gamma(k)$ the autocovariance function of a second order stationary process. Show that the function $\gamma(k)$ is positive semi-definite, i.e.

$$\sum_{s=1}^{N}\sum_{t=1}^{N} a_s a_t \gamma(s-t) \geq 0$$

for all positive integers N and for all vectors $(a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$.
Hint: Start with the left side of the inequality and replace the autocovariance by an expected value.

Question 6. Take y_t a second order stationary ARMA process, $P(L)y_t = Q(L)\epsilon_t$, with the roots of $P(z)$ outside the unit circle. Denote by $\gamma(h)$ the autocovariance function of y_t.

1. Show that there exist constants $C > 0$ and S, $0 < S < 1$, such that $|\gamma(h)| \leq CS^{|h|},\ h \in \mathbb{Z}$.
2. From this, deduct that $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

Hint 1: At what rate do the coefficients of $(1 - \lambda_1 L)^{-1}$ decrease to zero if $|\lambda_1| < 1$?
Hint 2: What can yo say about the (maximum) rate at which the coefficients of $\Psi(L) = (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1}$ decrease to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$?

Question 7. Consider the following MA(1) process,

$$y_t = c + \epsilon_t + \theta\epsilon_{t-1}$$

where ϵ_t is a white noise. Show that the absolute value of the first autocorrelation of this model can't be greater than 0.5 and find the value of the parameters for which this upper bound is obtained.