Two Phase Flow

Analysis of two phase flow systems is complicated by the appearance and disappearance of phases, discontinuous fluid properties at phase boundaries, the relative motion of the phases and mass, energy and momentum transfer between the phases. Of particular interest is the void distribution in a two phase channel. An example of a subcooled boiling channel is illustrated below.

Prior to the incipient boiling point (or nucleation point) $z_n$, no voids exist and the flow characteristics are those of a single phase liquid. Between the incipient boiling point and the bubble departure point ($z_d$), voids appear on the heated surface but remain attached to the wall. As there is no flow of the vapor phase, the flow quality is zero and the void fraction is related to the mass fraction of the vapor phase by the static quality. After the bubble departure point, the flow quality is greater than zero and two phase flow exist with the liquid phase subcooled and the vapor phase either superheated or saturated. Below the bulk boiling point ($z_{bulk}$), the average fluid enthalpy is below that of a saturated liquid. The equilibrium quality is less than zero and models based on equilibrium quality alone would fail to predict the existence of voids in the channel. At the bulk boiling point, the average fluid enthalpy equals that of a saturated liquid, even though the liquid phase may still be subcooled. The equilibrium quality is zero and models based on equilibrium quality would begin to predict the existence of vapor in the channel. At the equilibrium point ($z_{eq}$), the liquid and vapor phases are both at saturation and the flow quality and equilibrium quality are the same. Note, the phases reach thermodynamic equilibrium when both phases are at saturation, not when the equilibrium quality is zero. To accurately describe the mass, energy and momentum distributions in a two phase system requires conservation equations for each phase.

Figure 1: Void Distribution in a Heated Channel (from Rust, *Nuclear Power Plant Engineering*)
Consider the arbitrary two-phase control volume illustrated below. For ease of representation we have assumed an annular flow regime, though the multi-phase flow equations we derive will be valid regardless of the flow regime.

![Figure 1: Two-Phase Control Volume](image)

In our derivation of the single phase flow equations, we considered two types of surfaces: (a) Flow or convective surfaces (i.e. those at \( z \) and \( z + \Delta z \)) and (b) fixed or wall surfaces. The convective surfaces provided for the transport of mass, energy and momentum through the normal convective flow of the fluid. The fixed or wall surfaces provided surface sources of heat and momentum (drag forces). In two-phase flow, the interface between the liquid and vapor phases provides an additional surface for both convective and surface sources of mass, energy and momentum transfer. In this derivation we assume phase change can occur at heated surfaces (\( \dot{\delta}_w^' \)) and at the interface between the liquid and vapor (\( \dot{\delta}_i^' \)), where \( \dot{\delta}^' \) represents the phase change rate per unit length. We define a total effective phase change rate \( \dot{\delta}^' = \dot{\delta}_i^' + \dot{\delta}_w^' \) which is the sum of the interfacial and wall components and adopt a sign convention such that \( \dot{\delta}^' \) is positive for evaporation and negative for condensation. No assumptions are made regarding the state of the individual phases. For example, the liquid may be subcooled and the vapor superheated. Under these conditions, heat transfer from the superheated vapor to the liquid/vapor interface could lead to evaporation or condensation, depending on the relative states of the two phases. As a result, \( \dot{\delta}_w^' \) and \( \dot{\delta}_i^' \) can have different signs, such as in subcooled boiling where evaporation occurs at the wall, but condensation occurs once the bubbles have been injected into the flow stream.

**Mass Conservation (Phasic Continuity Equation)**

The two-phase conservation equations are obtained by performing mass, energy and momentum balances on the individual phases. Consider first a mass balance on the liquid phase

\[
\frac{dM_l}{dt} = \sum_{in} \dot{m}_{l,\text{in}} - \sum_{\text{exit}} \dot{m}_{l,\text{exit}}
\]  

(1)
Energy Conservation

Recall, that energy conservation is simply a restatement of the first law

\[ \dot{Q} \sum_{in} \dot{m}_{in} \left( e_{i} + P_{in} u_{i} \right) = \frac{dM_{e}}{dt} \sum_{exit} \dot{m}_{exit} \left( e_{exit} + P_{exit} u_{exit} \right) \]  

where \( e = u + \frac{v^{2}}{2} + gH \)

and we have neglected the external work terms. We again consider a control volume around the liquid phase such that the first law is

\[ \frac{dM_{e}}{dt} \sum_{exit} \dot{m}_{exit} \left( e_{i} + P_{i} u_{i} \right) - \sum_{in} \dot{m}_{in} \left( e_{i} + P_{i} u_{i} \right) = \dot{Q}_{l} \]  

For:

\[ M_{l} = \alpha_{l} \rho_{l} \overline{A}_{l} \Delta z \]

\[ \dot{m}_{l} \left( e_{i} + P_{i} u_{i} \right)_{exit} = (\alpha_{l} \rho_{l} e_{i} A_{i})_{z+\Delta z} + (\alpha_{l} P_{l} u_{i} A_{i})_{z+\Delta z} + \dot{\delta} \Delta z (e_{i} + P_{l} u_{i})_{i} \]

\[ \dot{m}_{l} \left( e_{i} + P_{i} u_{i} \right)_{in} = (\alpha_{l} \rho_{l} e_{i} A_{i})_{z} + (\alpha_{l} P_{l} u_{i} A_{i})_{z} \]

\[ \dot{Q}_{l} = \dot{q}_{l}^{\text{Wall}} + \dot{q}_{l}^{\text{Interface}} \]

the liquid phase energy equation is

\[ \overline{A}_{x} \Delta z \frac{d(\alpha_{l} P_{l} e_{l})}{dt} = (\alpha_{l} \rho_{l} e_{l} A_{l})_{z+\Delta z} + (\alpha_{l} P_{l} u_{l} A_{l})_{z+\Delta z} - (\alpha_{l} \rho_{l} e_{l} A_{l})_{z} - (\alpha_{l} P_{l} u_{l} A_{l})_{z} = -\dot{\delta} \Delta z (e_{i} + P_{i} u_{i})_{i} + \dot{q}_{l}^{\text{Wall}} + \dot{q}_{l}^{\text{Interface}} \]  

If we again divide by \( \Delta z \) and take the limit as \( \Delta z \) goes to zero we obtain the one-dimensional, differential form of the liquid phase energy equation
\[ A_x \frac{\partial (\alpha_i \rho_i v_i)}{\partial t} + \frac{\partial}{\partial z}(\alpha_i \rho_i v_i A_z) + \frac{\partial}{\partial z}(\alpha_i P_i v_i A_z) = -\delta'(e_i + P_i v_i)_t + \dot{q'}_{w,i} + \dot{q'}_{i,t} \]  

(8)

and similarly for the vapor phase

\[ A_x \frac{\partial (\alpha_g \rho_g e_g)}{\partial t} + \frac{\partial}{\partial z}(\alpha_g \rho_g e_g v_g A_z) + \frac{\partial}{\partial z}(\alpha_g P_g v_g A_z) = -\delta'(e_g + P_g v_g)_t + \dot{q'}_{w,g} + \dot{q'}_{i,g} \]  

(9)

Momentum Conservation

We consider the arbitrary two-phase control volume illustrated below.

A momentum balance on the liquid phase gives

\[ \frac{1}{g_c} \frac{d(M_i v_i)}{dt} = \frac{1}{g_c} \sum_{in} (\dot{m}_i v_i)_i - \frac{1}{g_c} \sum_{exit} (\dot{m}_i v_i)_i + \sum (F_i) \]  

(10)

For:  

\[ M_i = \alpha_i \rho_i \bar{A}_i \Delta z \\
(\dot{m}_i v_i)_i = (\alpha_i \rho_i v_i A_i) \big|_{z+\Delta z} + \delta' \Delta z (v_i)_i \\
(\dot{m}_i v_i)_i = (\alpha_i \rho_i v_i A_i) \big|_{z} \]

The liquid phase momentum equation is

\[ \frac{\bar{A}_i}{g_c} \Delta z \frac{d(\alpha_i \rho_i v_i)}{dt} + \frac{1}{g_c} \left( \alpha_i \rho_i v_i A_i \right) \big|_{z+\Delta z} - \frac{1}{g_c} \left( \alpha_i \rho_i v_i A_i \right) \big|_{z} = -\frac{1}{g_c} \delta' \Delta z (v_i)_i + \sum (F_i) \]  

(11)

As in the single phase case, the forces acting on the fluid are (1) pressure forces, (2) viscous or drag forces, (3) weight forces and (4) mechanical forces (if present).

Pressure Forces

The pressure forces in the two-phase equations are similar to those in the single-phase equations and may be written
\[ P_l A_l \left|_z - P_l A_l \left|_{z+\Delta z} + \overline{P}_l \left( A_l \left|_{z+\Delta z} - A_l \left|_z \right) \right) \right. \] (12)

**Viscous Forces**

In addition to wall shear, an additional interfacial shear force acts at the liquid/vapor interface. We will assume a sign convention, such that the vapor phase is exerting a positive force on the liquid phase (and conversely, a negative force on the vapor phase). The shear force is then

\[-(\tau_w \overline{P}_w)_l \Delta z + \tau_l \overline{P}_l \Delta z \] (13)

where \( P_l \) is the interfacial perimeter and a function of the flow regime.

**Weight Force**

The weight force in two-phase flow is again similar to that in single-phase flow and given as

\[-\alpha_t \rho_t \overline{A}_t \Delta z \frac{g}{g_c} \sin \theta \] (14)

**Mechanical Forces**

If pumps are present in the flow system, then the pump force acting on the liquid phase is

\[ \Delta P_{pl} A_l \Delta z \] (15)

The liquid phase momentum equation is then

\[ \frac{\overline{A}_l \Delta z}{g_c} \frac{\partial \alpha_t \rho_v v_l}{\partial \alpha} + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_t \rho_v v_l A_l \right) \right|_z - \frac{1}{g_c} \frac{\partial}{\partial z} \left( \beta_t \delta^z v_l \right)_z + P_l A_l \right|_z - P_l A_l \right|_{z+\Delta z} + \overline{P}_l \left( A_l \left|_{z+\Delta z} - A_l \left|_z \right) \right. \] (16)

\[-(\tau_w \overline{P}_w)_l \Delta z + (\tau_l \overline{P}_l) \Delta z - \alpha_t \rho_t \overline{A}_t \Delta z \frac{g}{g_c} \sin \theta + \Delta P_{pl} A_l \Delta z \]

We again divide by \( \Delta z \) and take the limit as \( \Delta z \to 0 \) to give

\[ \frac{A_x}{g_c} \frac{\partial \alpha_t \rho_v v_l}{\partial \alpha} + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_t \rho_v v_l A_x \right) \right|_z - \frac{1}{g_c} \frac{\partial}{\partial z} \left( P_l A_l \right)_z + P_l \frac{\partial A_l}{\partial z} \right|_z \]

\[-(\tau_w \overline{P}_w)_l + (\tau_l \overline{P}_l) \Delta z - \alpha_t \rho_t \overline{A}_t \Delta z \frac{g}{g_c} \sin \theta + \Delta P_{pl} A_l \]

(17)

or by expanding the pressure derivative and recognizing \( A_l = \alpha_t A_x \)

\[ \frac{A_x}{g_c} \frac{\partial \alpha_t \rho_v v_l}{\partial \alpha} + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_t \rho_v v_l A_x \right) \right|_z - \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_t A_x \right)_z - \alpha_t A_x \frac{\partial P_l}{\partial z} \right|_z \]

\[-(\tau_w \overline{P}_w)_l + (\tau_l \overline{P}_l) \Delta z - \alpha_t \rho_t \overline{A}_t \Delta z \frac{g}{g_c} \sin \theta + \Delta P_{pl} \alpha_t A_x \]

(18)

Similarly for the vapor phase
\[
\frac{A_k}{g_c} \frac{\partial \alpha_g \rho_g v_g}{\partial t} + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_g \rho_g v_g v_g A_k \right) = \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_g \rho_g A_k \right) - \frac{\alpha_g}{g_c} \frac{\partial P_g}{\partial z} \\
- (\tau_{w_P} - \tau_{l_P}) - \alpha_g \rho_g A_k \frac{g}{g_c} \sin \theta + \Delta P' \alpha_g A_k
\]

Equations 18 and 19 are the one-dimensional form of the phasic momentum equations.

**Closure**

The phasic conservation equations include a number of phase dependent interfacial terms, i.e. \( \dot{\alpha}, \tau_{l_P}, \dot{q}_{l,g}, \dot{q}_{l,l}, \)
\( (e_l + P_l u_l), (e_g + P_g v_g), (v_l), \) and \( (v_g). \) In addition, we have phase/wall interaction terms such as \( \dot{q}_{w,l}, \dot{q}_{w,g}, \)
\( (\tau_{w_P}) \) and \( (\tau_{l_P}). \) If we assume that these terms can be correlated in terms of the remaining phasic variables, then the phasic conservation equations contain as unknowns

Unknowns \( \Rightarrow \alpha_k, \rho_k, u_k, v_k \) and \( P_k \) \( (k = \ell, g) \) = 10

The phasic conservation equations provide six equations, implying four more are required for closure. As in the single phase equations, equations of state are required for each phase, i.e.

\[
\rho_k = \rho_k(u_k, P_k) \quad k = \ell, g
\]

which gives then an total of eight equations. In addition, we have the volume constraint

\[
\sum_{k=\ell,g} \alpha_k = 1
\]

leaving one equation required for closure. The last equation relates the phase pressures, i.e. \( P_l = P_l(P_g) \), the simplest of which is simply \( P_l = P_g = P \).

The phasic equations are the most general form of the multi-phase conservation equations and represent the current state of the art in reactor safety analysis. Since there exist 6 equations (two each for mass, energy and momentum) models based on the phasic conservation equations are often called six equation models. The phasic equations explicitly represent the motion and thermodynamic state of each phase and allow for the modeling of non-homogenous (unequal phase velocities) and non-equilibrium systems. While considered to be the most general form of the multi-phase equations, the phasic equations present computational difficulties and contain interfacial interaction terms which are in general not well known. As a result, we will concentrate on a simpler set of multi-phase equations which eliminate some of these concerns.
Mixture Equations

Mixture equations represent global balances of mass, energy and momentum in a two-phase system, and are obtained by adding the phasic balance equations.

Mixture Mass

To obtain a total mass balance on the control volume, add the phasic mass balance equations to yield

\[ A_x \frac{\partial}{\partial t} (\alpha_i \rho_i + \alpha_g \rho_g) + \frac{\partial}{\partial z} \left( \alpha_i \rho_i v_i + \alpha_g \rho_g v_g \right) A_x = 0 \] (22)

Note the cancellation of the interfacial mass transfer term, as mass leaving the liquid phase due to evaporation appears in the vapor phase. The term \( \alpha_i \rho_i + \alpha_g \rho_g \) is the effective two-phase density \( \rho \). In addition, from previous discussions \( \alpha_i \rho_i v_i + \alpha_g \rho_g v_g \) is the total mass flux \( G \), such the mass balance equation can be written as

\[ A_x \frac{\partial \rho}{\partial t} + \frac{\partial G A_x}{\partial z} = 0. \] (23)

Notice, that the mixture mass equation is identical in form to the single phase mass equation.

Mixture Energy

As with the total or mixture mass equation, we can obtain a mixture energy equation by adding the phasic equations.

\[ A_x \frac{\partial}{\partial t} (\alpha_i \rho_i e_i + \alpha_g \rho_g e_g) + \frac{\partial}{\partial z} \left( \alpha_i \rho_i e_i v_i + \alpha_g \rho_g e_g v_g \right) A_x + \frac{\partial}{\partial z} \left( \alpha_i P_i v_i + \alpha_g P_g v_g \right) A_x = \delta'(e_g + P_g v_g) - \delta(e_i + P_i v_i) \] (24)

The sum of the wall heat transfer terms is simply \( \dot{q}_w = \dot{q}_{w,t} + \dot{q}_{w,g} \) or the total two-phase linear heat rate. The treatment of wall heat transfer under two-phase conditions was considered in our discussion of boiling regimes and boiling heat transfer and can be relatively easily correlated in terms of mixture parameters. Equation 24 contains interfacial terms that at first glance do not immediately cancel. To address these additional terms, consider an energy balance on the interface illustrated below. Note, that since the interface has no volume, we need not consider volumetric or accumulation (time derivative terms). An energy balance on the interface gives

\[ \dot{q}_{g,t} + \delta'(e_i + P_i v_i) = \dot{q}_{l,t} + \delta'(e_g + P_g v_g) \] (25)

or

\[ \dot{q}_{g,t} = \dot{q}_{l,t} + \delta(e_g + P_g v_g - e_i - P_i v_i) \] (26)

Since \( \dot{q}_{g,t} = -\dot{q}_{l,g} \) the interfacial terms cancel in the mixture energy equation leaving

\[ A_x \frac{\partial}{\partial t} (\alpha_i \rho_i e_i + \alpha_g \rho_g e_g) + \frac{\partial}{\partial z} \left( \alpha_i \rho_i e_i v_i + \alpha_g \rho_g e_g v_g \right) A_x + \frac{\partial}{\partial z} \left( \alpha_i P_i v_i + \alpha_g P_g v_g \right) A_x = \dot{q}'_w \] (27)
To further simplify the energy equation, recall that

\[ \alpha_i \rho_i v_i = G(1 - x) \]
\[ \alpha_g \rho_g v_g = Gx \]

such that the convective terms may be written

\[ \alpha_i \rho_i e_i v_i + \alpha_g \rho_g e_g v_g + \alpha_i P_i v_i + \alpha_g P_g v_g = G(1 - x)(e_i + P_i v_i) + Gx(e_g + P_g v_g) \]

\[ G(1 - x)(e_i + P_i v_i) + Gx(e_g + P_g v_g) = G\left\{(1 - x)\left(h_i + \frac{v_i^2}{2} + gH\right) + x\left(h_g + \frac{v_g^2}{2} + gH\right)\right\} \]

(28)

\[ = G\left\{h + (1 - x)\left(\frac{v_i^2}{2}\right) + x\left(\frac{v_g^2}{2}\right) + gH\right\} \]

where we have defined a general mixture enthalpy

\[ h = (1 - x)h_i + xh_g \]

(29)

which is valid whether the phases are saturated or not. In this formulation, \(x\) is the flow quality, and from Equation 29 may be expressed as

\[ x = \frac{h - h_i}{h_g - h_i} \]

(30)

Note, that if the phases are at equilibrium, then this reduces to

\[ x = \frac{h - h_{ij}}{h_{jg}} \]

(31)
which is by definition the equilibrium quality. This is consistent with earlier statements that the equilibrium and flow qualities are equal under saturation conditions.

The energy equation may then be written as

$$A_x \frac{\partial}{\partial t} \left( \alpha_i \rho_i e_i + \alpha_g \rho_g e_g \right) + \frac{\partial}{\partial z} \left( G \left( h + \frac{(1-x)v_i^2}{2} + x \frac{v_g^2}{2} + gH \right) \right) A_x = \dot{q}_w'$$

(32)

If we neglect kinetic and potential energy terms, then $e_i \equiv u_i$ and the energy equation is

$$A_x \frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial z} \{G h\} A_x = \dot{q}_w'$$

(33)

where we have defined $\rho u = \alpha_i \rho_i u_i + \alpha_g \rho_g u_g$. Equation 33 is identical in form to energy equations we have used in the past. Again, no assumptions have been made relative to the fluid state, such that the enthalpy in Equation 33 is our general mixture enthalpy.

Mixture Momentum

To obtain the mixture momentum equation, we add Equations 18 and 19 to get

$$\frac{A_x}{g_c} \frac{\partial}{\partial t} \left( \alpha_i \rho_i v_i + \alpha_g \rho_g v_g \right) + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_i \rho_i v_i v_i + \alpha_g \rho_g v_g v_g \right) A_x = \frac{1}{g_c} \delta (v_g - v_i) - \alpha_i A_x \frac{\partial P_i}{\partial z}$$

$$- \alpha_g A_x \frac{\partial P_g}{\partial z} - (\dot{\tau}_w P_w)_i - (\alpha_i \rho_i + \alpha_g \rho_g) A_x \frac{g}{g_c} \sin \theta + \left( \alpha_i \Delta P_{pi} + \alpha_g \Delta P_{pg} \right) A_x$$

(34)

For $G = \alpha_i \rho_i v_i + \alpha_g \rho_g v_g$

$$\rho = \alpha_i \rho_i + \alpha_g \rho_g$$

and assuming equal phase pressures and equal phase velocities at the interface, the mixture momentum equation reduces to

$$\frac{A_x}{g_c} \frac{\partial}{\partial t} (G) + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_i \rho_i v_i + \alpha_g \rho_g v_g \right) A_x = -A_x \frac{\partial P}{\partial z}$$

$$- (\dot{\tau}_w P_w)_i - (\dot{\tau}_w P_w)_g - \rho A_x \frac{g}{g_c} \sin \theta + \left( \alpha_i \Delta P'_{pi} + \alpha_g \Delta P'_{pg} \right) A_x$$

(35)

We define the total two-phase viscous loss as

$$(\tau_w P_w)_{\phi} \equiv (\tau_w P_w)_i + (\tau_w P_w)_g$$

(36)

and the total two-phase pump Delta P as

$$\left( \Delta P'_{P} \right)_{\phi} \equiv \alpha_i \Delta P'_{pi} + \alpha_g \Delta P'_{pg}$$
\[
\frac{A_x}{g_c} \frac{\partial}{\partial z} (G) + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \alpha_l \rho_l v_l + \alpha_g \rho_g v_g \right) A_x = -A_x \frac{\partial P}{\partial z} \\
- (\tau_w P_w)_{2\phi} - \rho A_x \frac{g}{g_c} \sin \theta + \left( \Delta P'_p \right)_{2\phi} A_x \tag{37}
\]

The convective terms cannot be written solely in terms of mixture density and mass flux due to the unequal phase velocities. We can eliminate the phase velocities by noting

\[
\alpha_l \rho_l v_l = G(1-x) \Rightarrow v_l = \frac{G(1-x)}{\alpha_l \rho_l} \tag{38}
\]

\[
\alpha_g \rho_g v_g = Gx \Rightarrow v_g = \frac{Gx}{\alpha_g \rho_g} \tag{39}
\]

giving the final form of the mixture momentum equation

\[
\frac{A_x}{g_c} \frac{\partial}{\partial z} (G) + \frac{1}{g_c} \frac{\partial}{\partial z} \left( \frac{G^2(1-x)^2}{\alpha_l \rho_l} + \frac{G^2 x^2}{\alpha_g \rho_g} \right) A_x = -A_x \frac{\partial P}{\partial z} \\
- (\tau_w P_w)_{2\phi} - \rho A_x \frac{g}{g_c} \sin \theta + \left( \Delta P'_p \right)_{2\phi} A_x \tag{40}
\]

The mixture mass, energy and momentum equations will serve as the basis for further discussions of two-phase systems.

### Solution of the Mixture Equations

For simplicity, consider the steady-state form of the mixture equations

**Mass**

\[
\frac{\partial}{\partial z} (G) A_x = 0 \tag{41}
\]

**Energy**

\[
\frac{\partial}{\partial z} (Gh) A_x = g'_w \tag{42}
\]

**Momentum**

\[
\frac{1}{g_c} \frac{\partial}{\partial z} \left( G^2(1-x)^2 \alpha_l \rho_l + G^2 x^2 \alpha_g \rho_g \right) A_x = -A_x \frac{\partial P}{\partial z} \\
- (\tau_w P_w)_{2\phi} - \rho A_x \frac{g}{g_c} \sin \theta + \left( \Delta P'_p \right)_{2\phi} A_x \tag{43}
\]

supplemented by equations of state for each phase
State

\[ \rho_k = \rho_k (h_k, P) \quad k = \ell, g \]

where for convenience the state equation has been expressed in terms of enthalpy instead of internal energy. Again assuming the two-phase heat transfer and frictional terms can be correlated in terms of the remaining system variables, the number of unknowns is

Unknowns \( \Rightarrow \alpha_k, \rho_k, h_k, \rho, h, G, x, P \quad (k = \ell, g) = 11 \)

Since we now have only three conservation equations plus the two state equations, 6 additional equations or constraints on the fluid must be found. The volume constraint

\[ \sum_{k=\ell,g} \alpha_k = 1 \]

and definitions

\[ \sum_{k=\ell,g} \alpha_k \rho_k = \rho \]

\[ h = (1-x)h_\ell + x h_g \]

provide three additional equations, leaving three still required. At this point however, no additional equations are available which requires that we resort to assumptions regarding the motion and/or state of the individual phases. This information was lost when we added the phasic equations, which provided total balances on the fluid, but no information regarding the distribution of mass, energy and momentum among the phases.

void-Quality Relations

Recall in our earlier discussions of two-phase flow parameters, we derived the fundamental void-quality-slip relationship

\[ \alpha = \frac{1}{1 + \frac{1-x \rho_g}{x \rho_\ell} S} \quad (44) \]

where the slip ratio \( S = \frac{v_g}{v_\ell} \) represents the ratio of the vapor to liquid velocities. The void fraction is then implicitly a function of the phase velocities. If we assume homogeneous flow \( (S = 1) \), then Equation 44 reduces to

\[ \alpha = \frac{1}{1 + \frac{1-x \rho_g}{x \rho_\ell}} \quad (45) \]

providing the final equation necessary for solution of the mixture equations. The homogeneous flow assumption is overly restrictive however and does not provide a realistic representation of phase motion. As an alternative, semi-empirical void-quality relationships have been developed which implicitly account for the phase motion. One popular model of this type is the Zuber-Findlay model
\[
\alpha = \frac{1}{C_o \left[ 1 + \frac{1 - x}{x} \frac{\rho_g}{\rho_l} + \frac{\rho_g V_{\text{gj}}}{G_x} \right]} \quad (46)
\]

where \( C_o \) is the Concentration Parameter and \( V_{\text{gj}} \) is the Drift Velocity. \( C_o \) and \( V_{\text{gj}} \) are correlated parameters and are in general functions of the flow regime. However, for high pressure steam water flows

\[ C_o = 1.13 \]

\[ V_{\text{gj}} = 1.44 \left( \frac{\sigma_{gg} (\rho_l - \rho_g)}{\rho_l^2} \right)^0.74 \quad (47) \]

can be used regardless of flow regime. Note, that for \( C_o = 1 \) and \( V_{\text{gj}} = 0 \) the Zuber-Findlay model reduces to the homogeneous void-quality-slip model.

**Profile Fit Models**

Since no information is available regarding the relative states of the phases, we could assume the phases are at equilibrium, i.e.

\[ h_l = h_f (P) \]

\[ h_g = h_g (P) \]

Note, that this specification of the phase enthalpies will automatically produce saturation densities through the equations of state and forces the flow quality and equilibrium quality to be equal over the entire channel. The restriction on the phase states prevents modeling of subcooled boiling (from a fluid dynamics, not heat transfer standpoint) and other non-equilibrium effects and limits the applicability of the mixture model.

A number of empirical models have been developed to determine the flow quality in heated channels under nonequilibrium conditions. These “profile fit” models are generally functions of the equilibrium quality and fluid conditions at the bubble departure point. One such model is by Levy and has the form

\[ x = x_e - (x_e)_d \exp \left( \frac{x_e}{(x_e)_d} - 1 \right) \]

where \( x_e \) is the local equilibrium quality and \( (x_e)_d \) is the equilibrium quality at the bubble departure point, i.e.

\[ (x_e)_d = \frac{h_{id} - h_f}{h_{fg}} \]

where \( h_{id} \) is the liquid enthalpy at the bubble departure point and in single heated channels can be found from the simple energy balance

\[ h_{id} = h_{in} + \frac{1}{m} \int_0^{z_d} q'(z) \pi D dz \]
Note, that while \( x \) will always be greater than zero, the equilibrium quality will be negative for \( h < h_f \). The profile fit model allows for negative equilibrium quality.

The bubble departure point is also a correlated property. One correlation for the bubble departure point is the Saha-Zuber correlation

\[
h_f - h_{id} = \begin{cases} 
0.0022 \times q'(z_d) \times \frac{D_e C_{pe}}{k} & \text{Pe} < 70,000 \\
154 \times \frac{q'(z_d)}{G} & \text{Pe} > 70,000
\end{cases}
\]

where \( Pe = \frac{G D_e C_{pe}}{k} = Re \times Pr \) is the Peclet Number.

Since the heat flux is normally a nonlinear function of position, solution for the bubble departure point is in general iterative.

The void-quality relationship and the profile fit model add two of the three necessary equations for closure. Since no additional information is available, it is normally assumed that the vapor phase is saturated providing the final equation necessary for solution of the Mixture Equations.