Math 425 – Fall 2006 Name:

Test 2

This is an open book test. Please, state clearly the theorems you are using, justify your answers and write clearly to get credit for your work.

(1) (3 Pts) Let $F : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that there is a constant $M > 0$ and a neighborhood V of c such that $|f(x)| \leq M$ for all $x \in V \cap D$.

Since f is continuous at c, ~~[crossed out]~~ there is a $\delta > 0$

s.t. $|f(x) - f(c)| < 1$ for all $|x - c| < \delta$

Thus $-1 < f(x) - f(c) < 1$ and $f(c) - 1 < f(x) < 1 + f(c)$

for all $x \in V_\delta(c)$

Let $M = 1 + |f(c)|$

Then $|f(x)| \leq 1 + |f(c)|$ $\forall x \in V_\delta(c)$

(3) (2 Pts) Suppose that $f : I \in \mathbb{R}$, where I is an interval, is differentiable on I, and f' is bounded on I. Prove that f is a Lipschitz function on I.

By assumption, $|f'(x)| \leq M$ $\forall x \in I$, where $M > 0$.

By the Mean Value Thm., for each $x, y \in I$, $x < y$

there is a ~~[crossed out]~~ $c \in (x, y)$ s.t.

$$f(y) - f(x) = f'(c)\,(y - x)$$

Thus $|f(y) - f(x)| = |f'(c)|\,|y - x|$

Since $|f'(c)| \leq M$, then

$$|f(y) - f(x)| \leq M\,|y - x| \qquad \forall x, y \in I$$

This shows that f is a Lipschitz function on I

(2) (4 Pts) Suppose that f is continuous on $[a,b]$ and $\int_a^b f\,g = 0$ for every integrable function $g \in \mathcal{R}[a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Arguing by contradiction, suppose that there is a $c \in (a,b)$ s.t. $f(c) \neq 0$. (*)

Without loss of generality, suppose that $f(c) > 0$.

Then, $\exists\, V_\delta(c)$ where $f(c) > 0$.

In fact, since f is continuous at c, given $\varepsilon = \frac{1}{2} f(c)$

$\exists\, \delta > 0$ s.t. if $x \in V_\delta(c)$, then

$$|f(x) - f(c)| < \frac{1}{2} f(c)$$

Thus $f(x) > \frac{1}{2} f(c)$ $\quad \forall x \in V_\delta(c)$

Define $g(x) = \begin{cases} \frac{1}{2\delta} & \text{if } x \in V_\delta(c) \\ 0 & \text{if } x \notin V_\delta(c) \end{cases}$

$g \in \mathcal{R}[a,b]$ since g is a step function.

Then $\displaystyle\int_a^b f\,g = \int_{c-\delta}^{c+\delta} \frac{1}{2\delta} f(x)\,dx > \frac{2\delta}{2\delta}\,\frac{1}{2} f(c) = \frac{1}{2} f(c) > 0$

This is a contradiction. Thus $f = 0$ $\forall x \in [a,b]$

(*) The argument can be easily adjusted to the case where $c = a$ or $c = b$.

(4) (5 Pts) Let $f, g : D \to \mathbb{R}$. Show that:

(a) if f, g are uniformly continuous on D and bounded on D, then $f\,g$ is uniformly continuous;

(b) if f, g are uniformly continuous on D and D is a bounded set, then $f\,g$ is uniformly continuous (Hint: use part (a)).

(a) Since f, g are unif continuous, then, given any $\varepsilon > 0$, there is a $\delta > 0$ ($\delta = \delta(\varepsilon)$, but δ does not depend on x, y)

s.t. if $|x-y| < \delta$, then

$$|f(x) - f(y)| < \varepsilon, \quad |g(x) - g(y)| < \varepsilon$$

Observe that

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|$$

$$\leq |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)|$$

Since f, g are bounded on D, then $|f(x)|, |g(x)| \leq M \quad \forall x \in D$

Thus, for all $|x-y| < \delta$,

$$|f(x)g(x) - f(y)g(y)| \leq M\varepsilon + M\varepsilon = 2M\varepsilon$$

This shows that fg is unif. continuous on D.

(b) If f, g are uniformly continuous on D and D is bounded, then, by a RESULT proved in class, f, g are bounded on D

thus the uniform continuity of fg follows from part (a).