

Test 1

This is an open book test. Please, justify your answers and write clearly if you want credit for your work.

(1) (3Pts) Find all $x \in \mathbb{R}$ that satisfy $|x + 1| < |2x - 1|$.

Region:

$x < -1$	$-x - 1 < -2x + 1$ $x < 2$
$-1 < x < \frac{1}{2}$	$x + 1 < -2x + 1$ $3x < 0$ $x < 0$
$x > \frac{1}{2}$	$x + 1 < 2x - 1$ $x > 2$

\Rightarrow $x < 0$ & $x > 2$

The inequality is satisfied for:

(2) (4Pts) Give an example of:

(a) A bounded sequence that does not converge.

$$(-1)^n \quad |(-1)^n| \leq 1$$

(b) Two divergent sequences whose sum converges.

$$a_n = (-1)^n \quad b_n = (-1)^{n+1} \quad a_n + b_n = (-1)^n + (-1)^{n+1} = 0 \quad \forall n \geq 0$$

(c) A divergent sequence (x_n) for which the sequence of absolute values $(|x_n|)$ is convergent.

$$a_n = (-1)^n \quad |a_n| = 1$$

(d) A divergent sequence with a convergent subsequence (state both the sequence and the subsequence).

$$a_n = (-1)^n \quad a_{2n} = (-1)^{2n} = 1$$

(3) (3Pts) Let $a_1 = 1$, $a_{n+1} = \sqrt{2a_n + 3}$, $n \in \mathbb{N}$. Prove that (x_n) is convergent and find the limit. (Hint: use Monotone Convergence Theorem).

(a_n) IS BOUNDED: (I) $a_1 < 3$

(II) Suppose $a_n < 3$

(III) Then $a_{n+1} = \sqrt{2a_n + 3} < \sqrt{9} = 3$

(a_n) IS INCREASING: (I) $a_1 = 1 < a_2 = \sqrt{5}$

(II) Suppose $a_n < a_{n+1}$

(III) Then $a_{n+2} = \sqrt{2a_{n+1} + 3} > \sqrt{2a_n + 3} = a_{n+1}$

By the Monotone Convergence Thm., (a_n) IS CONVERGENT

Thus: $\lim a_{n+1} = \lim \sqrt{2a_n + 3}$. Let $\lim a_n = L$

$$L = \sqrt{2L + 3} \Rightarrow L^2 - 2L - 3 = 0 \quad (L=3) \text{ or } L = -1$$

The limit is $L=3$ ($L=-1$ is not acceptable since a_n)

(4) (2Pts) Compute the limit of $((1 + \frac{1}{n})^{n/2})$. (State the properties and/or theorems you use when you deduce your result).

We know that $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$

We also know that if $a_n \geq 0$ and $\lim a_n = L$,

then $\sqrt{a_n}$ converges and $\lim \sqrt{a_n} = \sqrt{L}$.

Thus
$$\lim (1 + \frac{1}{n})^{\frac{n}{2}} = \left(\lim (1 + \frac{1}{n})^n \right)^{1/2} = e^{1/2}$$

(5) (3Pts) Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, the $(x_n y_n)$ is also a Cauchy sequence. Justify carefully all your steps.

Since $(x_n), (y_n)$ are Cauchy, then they are bounded.

$$\text{let } |x_n| \leq M, |y_n| \leq M \quad \forall n$$

Since $(x_n), (y_n)$ are Cauchy, then given any ε

$$\exists N \text{ st. } |x_n - x_m| < \frac{\varepsilon}{2M}$$

$$|y_n - y_m| < \frac{\varepsilon}{2M} \quad \forall n, m > N$$

It follows that

$$\begin{aligned} |x_n y_n - x_m y_m| &\leq |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n| |y_n - y_m| + |y_m| |x_n - x_m| \\ &< M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} < \varepsilon \quad \forall n, m > N \end{aligned}$$

Thus $(x_n y_n)$ is a Cauchy sequence