2.1 Algebraic Properties

On the set $\mathbb{R}$, we define addition and multiplication with these properties (see textbook).

Order Properties

There is a subset of $\mathbb{R}$, say $\mathbb{P}$, of positive real numbers satisfying:

(i) $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.
(ii) $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.
(iii) If $a \in \mathbb{P}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \quad a = 0, \quad -a \in \mathbb{P}$$

If $a \in \mathbb{P}$, we say $a$ is a positive real number. [RK 0 is not a positive real]

It follows that we can order real numbers, by using inequalities (see textbook).

RK No smallest positive real number exists. If $a > 0$, then $\frac{1}{2}a > 0$.

Thus we can always find a positive real smaller than any given one.

Arithmetic-Geometric Mean Inequality

$$\sqrt{ab} \leq \frac{1}{2}(a + b)$$

For all $a, b \in \mathbb{P}$

Equality occurs if $a = b$.

Proof: $a < (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$

$$\Rightarrow a + b > 2\sqrt{ab}$$

$$\Rightarrow \frac{1}{2}(a + b) > \sqrt{ab}$$

Equality is obtained by observing that if

$$\frac{1}{2}(a + b) = \sqrt{ab} \Rightarrow (a + b)^2 = 4ab$$

$$\Rightarrow a^2 - 2ab + b^2 = 0$$

$$\Rightarrow (a - b)^2 = 0$$

$$\Rightarrow a = b$$
2.2 Absolute Value

Def: Let \( a \in \mathbb{R} \). The absolute value of \( a \), denoted by \( |a| \), is

\[
|a| = \begin{cases} 
  a & \text{if } a > 0 \\
  0 & \text{if } a = 0 \\
  -a & \text{if } a < 0 
\end{cases}
\]

Ex. If \( a = 3 \), then \( |3| = 3 \)
If \( a = -\sqrt{2} \), then \( |-\sqrt{2}| = \sqrt{2} \)

Properties

- \( |ab| = |a||b| \)
- \( a^2 = |a|^2 \)

For \( a \geq 0 \), then \( |a| \leq c \iff -c \leq a \leq c \)

In fact, \( |a| \leq c \iff \forall a \leq c \land -a \leq c \)

Combining these discussion: \(-c \leq a \leq c\)

- \(-|a| \leq a \leq |a|\)

(Triangle Inequality) \( |a+b| \leq |a| + |b| \)

In fact, \(-|a| \leq a \leq |a| \land -|b| \leq b \leq |b| \)

Then \(-(|a|+|b|) \leq a+b \leq (|a|+|b|) \)

Thus \( |a+b| \leq (|a|+|b|) \)

- \( |a-b| \leq |a| + |b| \)

- \( |a| - |b| \leq |a+b| \)

In fact, \( |ab| = |a-b| + |b| \leq |a-b| + |b| \)

\[
\implies |a-b| \geq |a|-|b| \\
|b| = 1 \frac{b}{b} - a \leq 1(b-a) + |a| \\
\implies |a-b| \geq 1b - (a) \\
\text{Thus} \quad |a-b| \leq |a|-|b| \leq |a-b| \\
\implies |a| - |b| \leq |a-b| \]
\[
\text{Compute the set of the solutions of } \\
13x - 5 < 7
\]

Solution
\[
\Rightarrow 3x - 5 < 7 \\
\Rightarrow 3x < 12 \\
\Rightarrow x < 4
\]
\[
\Rightarrow 3x > -2 \\
\Rightarrow x > -\frac{2}{3}
\]

In general:

Given \( a \in \mathbb{R} \), \( R > 0 \)

\( V_R(a) = \{ x \in \mathbb{R} : |x - a| < R \} \)

is the \( R \)-neighborhood of \( a \) given by

\[
V_R(a) = (a - R, a + R)
\]

Ex
\[
\text{Let } f(x) = \frac{2x^2 + 3x + 1}{2x - 1} \quad x \in (2, 3)
\]

Find \( \text{st. } |f(x)| \leq 20 \text{ for } x \in (2, 3) \)

Solution

\[
|2x^2 + 3x + 1| \leq 2|x|^2 + 3|x| + 1 \leq 2 \cdot 9 + 3 \cdot 3 + 1 = 28
\]

\[
|2x - 1| \geq 8
\]

Thus

\[
|f(x)| \leq \frac{28}{3}
\]

The absolute value is used to measure the distance between \( a, b \in \mathbb{R} \).

\[
d(a, b) = |a - b|
\]

\[
-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3
\]

\[
d(-1, 2) = |-1 - 2| = 3
\]

For \( a \in \mathbb{R} \),

\[
d(x, a) = |x - a| \text{ measure the distance between } x \in \mathbb{R}
\]
Completeness Property of \( \mathbb{R} \).

\( \mathbb{R} \) is a complete ordered field.

\[ \text{def} \quad \text{let} \quad S \ni x \quad \text{(c) } S \text{ is bounded above if } \exists u \in \mathbb{R} : \forall x \in S, x \leq u \text{ (d) } S \text{ is bounded below if } \exists w \in \mathbb{R} : \forall x \in S, w \leq x \text{ } \]

Observe that \( \mathbb{Q} \) has also all algebraic prop. However, \( \mathbb{R} \) has the additional prop. \( \mathbb{R} \) allows us to define limit operations.

(c) \( S \) is bounded above if \( \exists u \in \mathbb{R} : \forall x \in S, x \leq u \)

(b) \( S \) is bounded below if \( \exists w \in \mathbb{R} : \forall x \in S, w \leq x \)

(c) \( S \) is bounded if it has both bounds above and below. Otherwise, \( S \) is unbounded.

\[ \inf S \quad S \quad \sup S \quad \rightarrow \quad \mathbb{R} \]

\[ \exists x : x \geq 0 \] \(
\text{This set is bounded below } 0 \text{. (But also } -1, -2 \ldots \text{)}
\]

\[ \exists x : x > 0 \] \(
\text{This set is unbounded.}
\]

\[ \text{def} \quad \text{let} \quad S \ni x \quad \text{(a) } S \text{ is bounded above, then } u = \sup S \text{ (supremum of } S \text{ if least upper bound) } \]

\[ \text{inf \quad S \ni x \quad \text{(b) } S \text{ is bounded below, then } w = \inf S \text{ (infimum of } S \text{ if greatest lower bound) } \]

Not every set \( S \) has a sup or inf.

\[ \text{If an inf or sup exists, they are unique.} \]

\[ \exists x : x > 0 \quad \text{This set has no sup } S \text{. (In fact, } \inf S = 0 \text{ and } \sup S \text{ do not exist)} \]

\[ \exists x : x > 0 \quad \text{This set has no upper bound) \]

\[ \inf S = 0, \quad 0 \]

\[ \text{Rk} \quad \text{For a set } S \text{ there are a few possibilities} \]

\[ \text{(1) } S \text{ has both sup and inf} \]

\[ \text{(2) } S \text{ has sup but no inf} \]

\[ \text{(3) } S \text{ has inf but no sup} \]

\[ \text{These cases lead to no inf} \]

\[ \text{Ex} \quad S = \{ x : x > 0 \} \quad \text{(but } 0 \in S \} \]

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Properties
Let \( S \subseteq \mathbb{R} \) be nonempty.

- \( u = \sup S \iff \begin{align*}
(1) & \quad \forall v \in S, v \leq u \\
(2) & \quad \text{if } v < u, \exists s \in S : v < s
\end{align*} \)

- Let \( u \) be an upper bound of \( S \), then
\[ u = \sup S \iff \text{For each } \varepsilon > 0, \exists s \in S : u - \varepsilon < s \]

**Proof**

\((\Rightarrow)\) Let \( u = \sup S \) and \( \varepsilon > 0 \).

Since \( u - \varepsilon < u \), \( u - \varepsilon \) is not an upper bound.

Then \( \exists s \in S : s > u - \varepsilon \), thus \( u - \varepsilon < s \).

\((\Leftarrow)\) Let \( u \) be an upper bound of \( S \) and \( \varepsilon > 0 \).

Let \( \varepsilon = u - w \). Then \( \exists s \in S \) s.t. \( u = u - \varepsilon < s \).

Thus \( u \) is not a sup. Thus \( u = \sup S \).

- If \( S \neq \emptyset \) has finitely many elements, then \( \sup S = \max S \) (largest element) and \( \inf S = \min S \) (smallest element).

Completeness Property of \( \mathbb{R} \)
Every nonempty set of \( \mathbb{R} \) that has an upper bound, also has a supremum in \( \mathbb{R} \).

This property does not follow from the algebraic prop. of \( \mathbb{R} \).

This property is not true for \( \mathbb{Q} \).

2.4 Applications of the Supremum Property

Let \( S \subseteq \mathbb{R} \), \( S \neq \emptyset \), \( x \in \mathbb{R} \).
Define \( a + S = \{ a + s : s \in S \} \)

Claim: \( \sup (a + S) = a + \sup S \).

**Proof**

Let \( u = \sup S \). Then \( x \leq u \) \( \forall x \in S \).

Then \( a + x \leq a + u \) \( \forall x \in S \).

This shows that \( \sup (a + S) \leq a + \sup S \).

Def

Let \( f : D \rightarrow \mathbb{R} \).
\( f \) is bounded above if \( \text{the set } f(D) = \{ f(x) : x \in D \} \) is bounded above.
\( f \) is bounded below if \( \text{the set } f(D) \) is bounded below.
\( f \) is bounded if it is both above and below.

Ex

\[ f(x) = x^2 \quad \text{for } -1 \leq x \leq 1 \]

\[ f(D) = \{ x^2 : -1 \leq x \leq 1 \} \]

\[ f(x) \leq 1 \quad \forall x \in D \]

\[ f(x) \geq 0 \quad \forall x \in D \]

If \( f(x) \leq g(y) \) \( \forall x, y \in D \), then \( \sup f(D) \leq \inf g(D) \).
Archimedeas Property

If \( a \in \mathbb{R} \), then \( \exists n \in \mathbb{N} \) s.t. \( x < ny \)

This shows that the natural numbers are not full.

It follows that:

1. \( \inf \{ \frac{1}{n} : n \in \mathbb{N} \} = 0 \)

(proofs in book)

2. If \( x > 0 \), then \( \exists n \in \mathbb{N} \) s.t. \( \frac{1}{n} < x \)

3. If \( x > 0 \), then \( \exists n \in \mathbb{N} \) s.t. \( nx \leq y \)

One of the consequences of density properties is the existence of irrational numbers. This was discovered by Greek math. already.

We have shown that \( \mathbb{Q} \) is countable. However, \( \mathbb{R} \) is uncountable.

However, \( \mathbb{Q} \) is dense in \( \mathbb{R} \), in the sense that given any two real numbers, there is a real number between them:

**Theorem (Density Thm)**: If \( x, y \in \mathbb{R} \) with \( x < y \), then \( \exists \frac{p}{q} \in \mathbb{Q} : x < \frac{p}{q} < y \)

**Proof**

Let \( x > 0 \) (the argument is the same for \( x < 0 \)).

Then \( y - x > 0 \) and, thus, \( \exists n \in \mathbb{N} \) s.t. \( y - x > \frac{1}{n} \).

This implies that \( y - nx > 0 \) \( \Rightarrow \) \( nx + 1 < y \)

It follows that \( m = nx + 1 \), thus:

Take \( m \in \mathbb{N} \) s.t. \( m - 1 \leq nx < m \)

Then \( m - 1 < nx < m \)

Thus \( m - 1 < nx + 1 < y \)

Thus \( x < \frac{m}{n} < y \)

Set \( r = \frac{m}{n} \)

**Corollary**: If \( x, y \in \mathbb{R} \), with \( x < y \), then \( \exists \frac{p}{q} \in \mathbb{R} \setminus \mathbb{Q} \) s.t. \( x < \frac{p}{q} < y \)

**Proof**

Use the above with

\[
\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}, \text{ then } x < \frac{\sqrt{2}}{r} < y
\]

Thus \( \frac{\sqrt{2}}{r} \in \mathbb{R} \setminus \mathbb{Q} \)