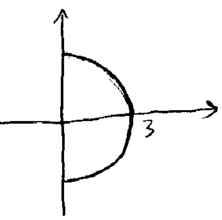
Math 242 Sec. 8 – Fall 2007 Name: SOLUTION

TEST #4

Only one page of notes allowed. No calculators. Please, justify your answers and write clearly if you want credit for your work.

(1) [5 Pts] Evaluate the mass and center of mass of a thin wire in the shape of the semicircle $x^2 + y^2 = 9$, $x \geq 0$, if the density function is a constant k.



Use parametrization $\underline{r}(t) = (3\cos t, 3\sin t)$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ [1Pt]

$\underline{r}'(t) = 3(-\sin t, \cos t)$ $\quad |\underline{r}'(t)| = 3$ [1Pt]

$m = \int_C k\, ds = \int_{-\frac{\pi}{2}}^{\pi/2} k\, |\underline{r}'(t)|\, dt = 3k\pi$ [1Pt]

$\bar{x} = \frac{1}{m} \int_C x\, ds = \frac{1}{3k\pi} \int_{-\frac{\pi}{2}}^{\pi/2} k\,(3\cos t)\, 3\, dt = \frac{3}{\pi} \sin t \Big|_{-\frac{\pi}{2}}^{\pi/2} = \frac{6}{\pi}$ [1Pt]

$\bar{y} = \frac{1}{m} \int_C y\, ds = \frac{1}{m} \int_{-\frac{\pi}{2}}^{\pi/2} k\,(3\sin t)\, 3\, dt = 0 \quad \Rightarrow \quad (\bar{x}, \bar{y}) = (\frac{6}{\pi}, 0)$ [1Pt]

(2) [5 Pts.] Let $\mathbf{F}(x,y) = (ye^{xy} + \cos y)\,\mathbf{i} + (xe^{xy} - x\sin y - 1)\,\mathbf{j}$ be a force field.

(a) Verify that $\mathbf{F}$ is a conservative vector field.

(b) Compute the work done by the force $\mathbf{F}$ to move an object along the curve $\mathbf{r}(t) = (1 + \sin t, \cos t)$, for $0 \leq t \leq \pi/2$.

(a) $\frac{\partial}{\partial y}(ye^{xy} + \cos y) = e^{xy} + xye^{xy} - \sin y$

$\frac{\partial}{\partial x}(xe^{xy} - x\sin y - 1) = e^{xy} + xye^{xy} - \sin y$

$\Rightarrow$ $\underline{F}$ is conservative, so $\underline{F} = \nabla f$ [1Pt]

(b) We compute f.

$f_x = ye^{xy} + \cos y \quad \Rightarrow \quad f(x,y) = \int (ye^{xy} + \cos y)\, dx = e^{xy} + x\cos y + K(y)$

$f_y(x,y) = xe^{xy} - x\sin y + K'(y)$

$= xe^{xy} - x\sin y - 1 \quad \Rightarrow \quad K(y) = -y + C$

Thus $\boxed{f(x,y) = e^{xy} + x\cos y - y}$ [3Pts]

By the F.T. for Line Integrals,

$W = f(\underline{r}(\frac{\pi}{2})) - f(\underline{r}(0)) = f(2,0) - f(1,1)$

$= (1+2) - (e + \cos(1) - 1) = \boxed{4 - e - \cos(1)}$ [1Pt]

(3) [5 Pts.] Evaluate the surface integral

$$\iint_S y\,dS,$$

where S is the helicoid $\mathbf{r}(u,v) = (u\cos v, u\sin v, v)$, for $0 \le u \le 1$ and $0 \le v \le \pi$.

$\underline{r}_u = (\cos v, \sin v, 0) \qquad \underline{r}_v = (-u\sin v, u\cos v, 1)$ [1 Pt]

$\underline{r}_u \times \underline{r}_v = (\sin v, -\cos v, u) \qquad |\underline{r}_u \times \underline{r}_v| = \sqrt{1+u^2}$ [1 Pt]

$$\iint_S y\,dS = \int_0^{\pi}\int_0^1 (u\sin v)\sqrt{1+u^2}\,du\,dv = \int_0^{\pi}\sin v\,dv \int_0^1 \sqrt{1+u^2}\,u\,du =$$ [2 Pts]

$$= -\cos v\Big|_0^{\pi}\ \frac{1}{2}\int_1^2 \sqrt{y}\,dy = 2\,\frac{1}{2}\,\frac{2}{3}\,y^{3/2}\Big|_1^2 = \boxed{\frac{2}{3}\left(2^{3/2}-1\right)}$$ (1 Pt)

$(y = 1+u^2)$

(4) [5 Pts.] Evaluate the flux

$$\Phi = \iint_S \mathbf{F}\cdot d\mathbf{S},$$

where $\mathbf{F}(x,y,z) = (2x, -y, 2z)$ and S is the part of the sphere $x^2+y^2+z^2 = 9$ in the first octant with upward orientation.

$\mathbf{r}(u,v) = (3\sin u\cos v, 3\sin u\sin v, 3\cos u) \qquad 0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le \frac{\pi}{2}$ [1 Pt]

$\underline{r}_u \times \underline{r}_v = 9(\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u)$ [1 Pt]

$\underline{F}(\underline{r}(u,v)) = (6\sin u\cos v, -3\sin u\sin v, 6\cos u)$ [1 Pt]

Thus

$$\Phi = 27\iint_D 2\sin^3 u\cos^2 v - \sin^3 u\sin^2 v + 2\sin u\cos^2 u\,du\,dv$$ (2 Pts)

$$= 27\left[\int_0^{\pi/2} 2\cos^2 v\,dv \int_0^{\pi/2}\sin^3 u\,du - \int_0^{\pi/2}\sin^2 v\,dv\int_0^{\pi/2}\sin^3 u\,du + \pi\int_0^{\pi/2}\cos^2 u\sin u\,du\right]$$

$$= 27\left[2\cdot\frac{\pi}{2}\cdot\frac{2}{3} - \frac{\pi}{2}\cdot\frac{2}{3} + \pi\,\frac{1}{3}\right] = 27\,\frac{\pi}{2}$$

In fact

$$\int_0^{\pi/2}\sin^3 u\,du = \int_0^{\pi/2}(1-\cos^2 u)\sin u\,du = -\int_1^0 (1-y^2)\,dy = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^{\pi/2}\cos^2 u\sin u\,du = -\frac{\cos^3 u}{3}\Big|_0^{\pi/2} = \frac{1}{3}$$

Alternative Solution

$$\Phi = \iiint \operatorname{div}\mathbf{F} = 3\iiint_E dV$$

$$= 3\,\frac{1}{8}\,\frac{4}{3}\pi\,3^3$$

$$= \frac{27\pi}{2}$$