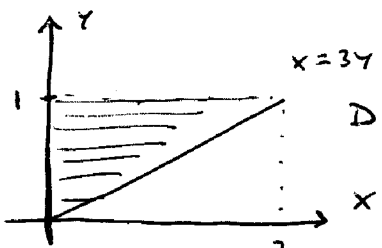
Math 242 Sec. 8 – Fall 2007 Name: (SOLUTION)

TEST #3

Only one double-sided page of notes allowed. No calculators. Please, justify your answers and write clearly if you want credit for your work.

(1) [3 Pts] In order to solve the following integral, you need to reverse the order of integration. Without solving the integral, write the integral that you obtain when the order of integration is reversed. Sketch the region of integration.

$$\int_0^1 \int_0^{3y} e^{x^2}\, dx\, dy.$$



(1 Pt)

$$D = \left\{ 0 \le x \le 3,\ \frac{x}{3} \le y \le 1 \right\}$$

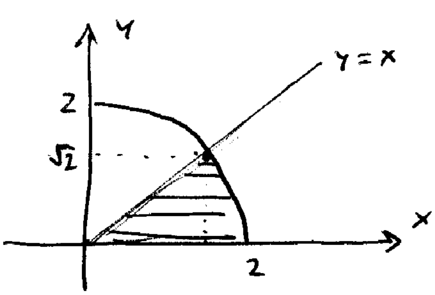
$$I = \int_0^3 \int_{x/3}^1 e^{x^2}\, dy\, dx$$

(2 Pts)

(2) [4 Pts.] Use polar coordinates to evaluate:

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy.$$

Sketch the region of integration.



(1 Pt)

$$D = \left\{ 0 \le r \le 2,\ 0 \le \vartheta \le \frac{\pi}{4} \right\}$$

$$I = \int_0^{\pi/4} \int_0^2 \frac{1}{1+r^2}\, r\, dr\, d\vartheta$$

(2 Pts)

$$= \frac{\pi}{4}\, \frac{1}{2} \int_1^5 \frac{1}{u}\, du = \frac{\pi}{8} \ln(5)$$

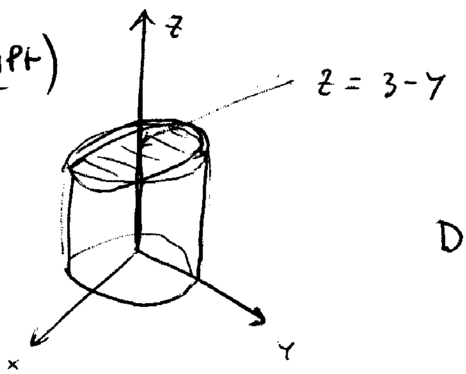
(1 Pt)

Set $u = 1 + r^2$

$du = 2r\, dr$

(3) [4 Pts] Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $y+z=3$. Sketch the solid.

(1Pt)



$$V = \iint_D (3-y)\, dA$$

Use polar coordinates in D

$$D = \{x^2+y^2 \le 4\} = \{(r,\theta): 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$ (1Pt)

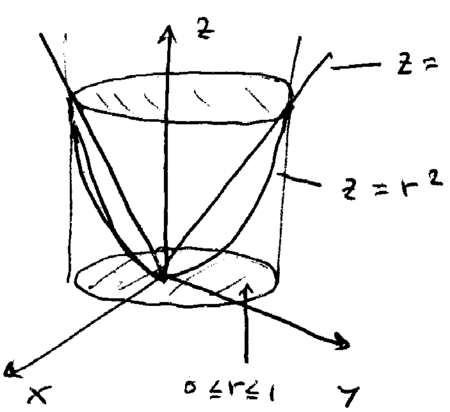
$$V = \int_0^{2\pi}\int_0^2 (3 - r\sin\theta)\, r\, dr\, d\theta =$$ (1Pt)

$$= \int_0^{2\pi}\int_0^2 (3r - \cancel{r^2\sin\theta})\, dr\, d\theta = 3\ 2\pi\ \frac{r^2}{2}\Big|_0^2 = \boxed{12\pi}$$ (1Pt)

(4) [5 Pts] Compute the integral

$$\iiint_E y^2\, dV,$$

where E is the solid above the paraboloid $z=x^2+y^2$ and below the half cone $z=\sqrt{x^2+y^2}$. Sketch the solid.



(1Pt)

Use cylindrical coordinates

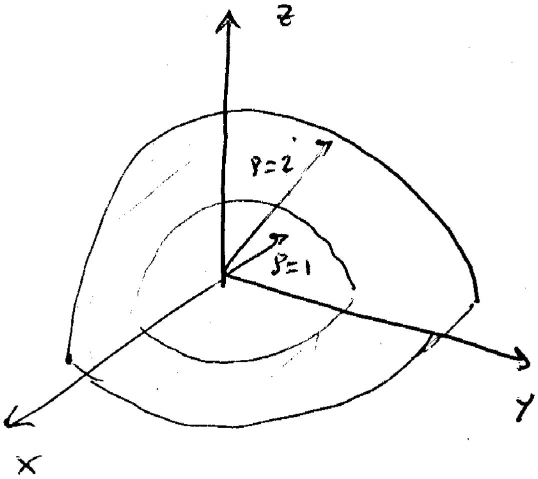
$$E = \{0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le r\}$$ (1Pt)

$$I = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r^2\sin^2\theta\, dz\, r\, dr\, d\theta$$ (2Ps)

$$= \int_0^{2\pi}\sin^2\theta\, d\theta \int_0^1 r^3(r-r^2)\, dr =$$

$$= \int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\, d\theta \left(\frac{1}{5}-\frac{1}{6}\right) = \boxed{\frac{\pi}{30}}$$ (1Pt)

(5) [4 Pts.] Find the mass of solid that occupies the region E bounded between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$, for $z \geq 0$, whose density is $\rho(x,y,z) = \frac{z}{\sqrt{1+(x^2+y^2+z^2)^2}}$.



Use spherical coordinates

$$E = \left\{ 1 \leq \rho \leq 2 ,\ 0 \leq \theta \leq 2\pi ,\ 0 \leq \varphi \leq \frac{\pi}{2} \right\}$$

$$V = \iiint_E \frac{z}{\sqrt{1+(x^2+y^2+z^2)^2}}\, dV =$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \frac{\rho\cos\varphi}{\sqrt{1+\rho^4}}\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

(3Pts)

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi\cos\varphi\, d\varphi \int_1^2 \frac{\rho^3}{\sqrt{1+\rho^4}}\, d\rho$$

(1Pt)

$$= 2\pi \left.\frac{\sin^2\varphi}{2}\right|_0^{\pi/2} \frac{1}{4}\int_2^{17} \frac{1}{\sqrt{u}}\, du = 2\pi\cdot\frac{1}{2}\,\frac{2}{4}\left(\sqrt{17}-\sqrt{2}\right)$$

$$= \frac{\pi}{2}\left(\sqrt{17}-\sqrt{2}\right)$$

$u = 1+\rho^4$

$du = 4\rho^3\, d\rho$