Math 242 Sec. 8 – Fall 2007 Name: SOLUTION

TEST #2

Only one page of notes allowed. No calculators. Please, justify your answers and write clearly if you want credit for your work.

(1) [4 Pts.] Find the following limit, if it exists, or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^6+2y^3}}$$

Set $f(x,y) = \dfrac{xy}{\sqrt{x^6+2y^3}}$

- Choose the path $x=0$. Then $f(0,y) = \dfrac{0}{\sqrt{2y^3}}$ and $\displaystyle\lim_{(0,y)\to 0} f(x,y) = \lim_{y\to 0} \frac{0}{\sqrt{2y^3}} = 0$ [1 Pt]
- Next set $y = x^2$. Then $f(x,x^2) = \dfrac{x^3}{\sqrt{x^6+2x^6}} = \dfrac{x^3}{\sqrt{3x^6}} = \dfrac{1}{\sqrt{3}}$

We have: $\displaystyle\lim_{(x^2,x)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3}{\sqrt{3x^6}} = \frac{1}{\sqrt{3}}$ [2 Pt] (For distinct path solution $\neq 0$)

Since different paths lead to different limits, then the LIMIT DOES NOT EXIS [1 Pt] (Conclusions)

(2) [4 Pts.] Find an equation of the tangent plane to the surface $z = \ln(x^2 - 2y)$ at the point $(3, 4, 0)$.

Define $F(x,y,z) = \ln(x^2-2y) - z = 0$

$$\nabla F(x,y,z) = \left(\frac{2x}{x^2-2y}, -\frac{2}{x^2-2y}, -1\right)$$ [2 Pt]

$$\nabla F(3,4,0) = (6, -2, -1)$$ [1 Pt]

TANGENT PLANE: $6(x-3) - 2(y-4) - z = 0$

$$\boxed{6x - 2y - z - 10 = 0}$$ [1 Pt]

(3) [4 Pts] The radius of a circular cone is increasing at a rate of 3 in/s while its height is decreasing at a rate of 2 in/s. At what rate is the volume of the cone changing when the radius is 100 inches and the height is 200 inches?

$$V(r,h) = \frac{\pi r^2 h}{3}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2\pi r h}{3}\frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$ [2 Pts]

At $r = 100$, $h = 200$, $\frac{dr}{dt} = 3$, $\frac{dh}{dt} = -2$, we have:

$$\frac{dV}{dt} = \frac{2\pi\, 100 \cdot 200}{3} \cdot 3 - \frac{\pi\, 100^2}{3}\, 2 = 4\pi\, 10^4 - \frac{2}{3}\pi\, 10^4 = \boxed{\frac{\pi}{3} 10^5 \left[\frac{in^3}{s}\right]}$$ [2 Pts]

(4) [4 Pts.] The temperature T on a planar lamina is described by

$$T(x,y) = \frac{x}{x^2 + 3y}.$$

Find: (a) In which direction is the temperature changing most rapidly at $P(2,1)$? (b)] What is the maximum rate of change of T at the point $P(2,1)$?

$$\begin{cases} T_x(x,y) = \dfrac{(x^2+3y) - 2x^2}{(x^2+3y)^2} = \dfrac{3y - x^2}{(x^2+3y)^2} \\ T_y(x,y) = -\dfrac{3x}{(x^2+3y)^2} \end{cases}$$ [1 Pt]

$$T_x(2,1) = -\frac{1}{49}, \quad T_y(2,1) = -\frac{6}{49}$$ [1 Pt]

(b) $$|\nabla T(2,1)| = \frac{1}{49}\sqrt{1+36} = \boxed{\frac{\sqrt{37}}{49}}$$ [1 Pt]

(a) $$u = \frac{\nabla T(2,1)}{|\nabla T(2,1)|} = \boxed{-\frac{1}{\sqrt{37}}(1,6)}$$ [1 Pt]

[4 Pts to find CRITICAL POINTS
1 Extra Point to determine loc MAX/MIN/SADDLE]

(5) Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = 5xy + x^2y + xy^2$.

$$\begin{cases} f_x(x,y) = 5y + 2xy + y^2 = y(5 + 2x + y) = 0 \\ f_y(x,y) = 5x + x^2 + 2xy = x(5 + x + 2y) = 0 \end{cases}$$

[1 Pt]

I eq. has zeros at $y = 0$, $y = -5 - 2x$

For $y = 0$, II eq. has zeros at $x = 0$

$5 + x = 0 \Rightarrow x = -5$

Hence $\boxed{(0,0),\ (-5,0) \text{ are critical points}}$

[3 Pt for the critical points]

For $y = -5 - 2x$, II eq. has zeros at $x = 0$

$5 + x + 2(-5 - 2x) = -5 - 3x = 0$

$\Rightarrow x = -\frac{5}{3}$

For $x = 0$, $y = -5$

For $x = -\frac{5}{3}$, $y = -5 + \frac{10}{3} = -\frac{5}{3}$

Hence $\boxed{(0,-5),\ \left(-\frac{5}{3}, -\frac{5}{3}\right) \text{ are critical points}}$

$f_{xx}(x,y) = 2y$, $f_{yy} = 2x$, $f_{xy} = 5 + 2x + 2y$

$D(x,y) = 4xy - (5 + 2x + 2y)^2$

$(0,0)$ $D(0,0) = -25 < 0 \Rightarrow$ SADDLE POINT

$(-5,0)$ $D(-5,0) = -25 < 0 \Rightarrow$ SADDLE POINT

$(0,-5)$ $D(0,-5) = -25 < 0 \Rightarrow$ SADDLE POINT

[1 Extra Point]

$\left(-\frac{5}{3}, -\frac{5}{3}\right)$ $D\left(-\frac{5}{3}, -\frac{5}{3}\right) = 4\frac{25}{9} - \left(5 - \frac{10}{3} - \frac{10}{3}\right)^2 = \frac{100}{9} - \frac{25}{9} > 0$

$f_{xx}\left(-\frac{5}{3}, -\frac{5}{3}\right) = -\frac{10}{3} < 0 \Rightarrow$ LOCAL MAXIMUM