Math 242 Sec. 12 – Fall 2006 Name: SOLUTION

TEST #4

Only one page of notes allowed. No calculators. Please, justify your answers and write clearly if you want credit for your work.

(1) [4 Pts] Evaluate

$$\int_C x^3\, z\, ds$$

where C is given by $x = 2\sin t, y = t, z = 2\cos t$, $0 \le t \le \pi/2$.

$\underline{r}(t) = (2\sin t,\ t,\ 2\cos t)$ [1 Pt]

$\underline{r}'(t) = (2\cos t,\ 1,\ -2\sin t)$

$|\underline{r}'(t)| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t} = \sqrt{5}$ [1 Pt]

$$\int_C x^3 z\, ds = \int_0^{\pi/2} 8\sin^3 t\ 2\cos t\ \sqrt{5}\, dt = 16\sqrt{5}\int_0^{\pi/2} \sin^3 t\, \cos t\, dt \qquad [1\ \text{Pt}]$$

$$= 16\sqrt{5}\ \frac{\sin^4 t}{4}\Big|_0^{\pi/2} = \boxed{4\sqrt{5}} \qquad [1\ \text{Pt}]$$

(2) [4 Pts.] Let $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2\, e^{xy})\,\mathbf{j}$ be a force field.

(a) Verify that $\mathbf{F}$ is a conservative vector field.

(b) Compute the work done by the force $\mathbf{F}$ to move an object along the curve $\mathbf{r}(t) = (1 + \sin t, \cos t)$, for $0 \le t \le \pi/2$.

(a) $P(x,y) = (1+xy)\,e^{xy}$, $Q(x,y) = e^y + x^2 e^{xy}$ [1 Pt]

$P_y(x,y) = 2x\,e^{xy} + x^2 y\, e^{xy} = Q_x(x,y)$. Thus $\boxed{\underline{F} \text{ is conservati}}$

(b) Need to find $f(x,y)$ s.t. $\underline{F}(x,y) = \nabla f(x,y)$

$f_y(x,y) = e^y + x^2 e^{xy}$

$f(x,y) = \int (e^y + x^2 e^{xy})\, dy = e^y + x e^{xy} + g(x)$ [2 Pt]

$f_x(x,y) = xy\, e^{xy} + e^{xy} + g'(x) = (1+xy)\, e^{xy} \Rightarrow g'(x) = 0$

$\Rightarrow g(x) = K$

$\boxed{f(x,y) = e^y + x e^{xy} + K}$

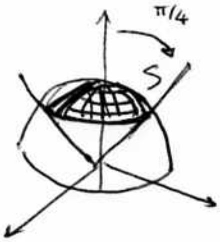
$W = f(2,0) - f(1,1) = (1+2) - (e+e) = \boxed{3 - 2e}$ [1 Pt]

(3) [5 Pts.] Evaluate

$$\iint_S xyz\, dS,$$

where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$, in the first octant.

Parametrize S as $\underline{r}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$ [1Pt]



Since we only consider the part above the cone, we have: [1Pt]

$0 \le u \le \pi/4$

$0 \le v \le \pi/2$ $\quad D = \{(u,v):\ 0 \le u \le \pi/4,\ 0 \le v \le \frac{\pi}{2}\}$

$|\underline{r}_u \times \underline{r}_v| = \sin u$

[2Pt] $$\iint_S xyz\, dS = \iint_D \sin^3 u \cos u \cos v \sin v \, dA = \int_0^{\pi/4} \sin^3 u \cos u \, du \int_0^{\pi/2} \sin v \cos v \, dv$$

$$= \left.\frac{\sin^4 u}{4}\right|_0^{\pi/4} \left.\frac{\sin^2 v}{2}\right|_0^{\pi/2} = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{32}$$ [1Pt]

(4) [2 Pts] Find a parametric representation for the part of the cylinder $x^2+z^2=1$ that lies between the planes $y=1$ and $y+x=3$.

Set $x = \cos t$

$z = \sin t$ $\quad$ Then $x^2+z^2=1$ $\quad 0 \le t \le 2\pi$

$y = u$ $\quad 1 \le y \le 3-x$

$1 \le u \le 3-\cos t$

Thus

$$\boxed{\begin{array}{c} \underline{r}(t,u) = (\cos t, u, \sin t) \\ 0 \le t \le 2\pi, \quad 1 \le u \le 3-\cos t \end{array}}$$ [1Pt] [1Pt]

(5) [5 Pts.] Evaluate the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = (-y, x, 2z)$ and S is the hemisphere $z = \sqrt{9 - x^2 - y^2}$ with upward orientation.

parametrization of S:

$$\underline{r}(u,v) = (3 \sin u \cos v,\ 3 \sin u \sin v,\ 3 \cos u) \qquad [1\text{ pt}]$$

$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi$$

$$\underline{r}_u \times \underline{r}_v = (9 \sin^2 u \cos v,\ 9 \sin^2 u \sin v,\ 9 \sin u \cos u)$$

$$\underline{F}(\underline{r}(u,v)) = (-3 \sin u \sin v,\ 3 \sin u \cos v,\ 6 \cos u) \qquad [1\text{pt}]$$

$$\underline{F}(\underline{r}(u,v)) \cdot \underline{r}_u \times \underline{r}_v = -27 \sin^3 u \sin v \cos v + 27 \sin^3 u \sin v \cos v + 54 \sin u \cos^2 u \qquad [1\text{pt}]$$

Thus

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_D \underline{F}(\underline{r}(u,v)) \cdot (\underline{r}_u \times \underline{r}_v)\, dA =$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 54 \cos^2 u \sin u \; du\, dv = \qquad [1\text{pt}]$$

$$= 2\pi \cdot 54 \left(-\frac{\cos^3 u}{3}\right)\Bigg|_0^{\pi/2} = 36\pi \qquad [1\text{pt}]$$

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Alternative solution

Using divergence Thm:

$$\iint_S \underline{F} \cdot d\underline{S} = \iiint_E 2\, dV = 2\, \frac{2\pi}{3}\, 3^3 = 36\pi$$

observing that $\iint_{S_0} \underline{F} \cdot d\underline{S} = 0$, where S_0 is the plane $z = 0$.

This must be justified as we did in class