Math 242 Sec. 4 – Fall 2004 Name: SOLUTION

TEST #4

One double-sided page of notes and a calculator or laptop with Maple are allowed. Please, justify your answers and write clearly if you want credit for your work.

(1) [3 Pts.] Let C be the arc of parabola $y = x^2$, for $0 \leq x \leq 1$. Evaluate the line integral:

$$\int_C x\, ds.$$

Parametrize C as $\begin{cases} x = x \\ y = x^2 \end{cases}$ $0 \leq x \leq 1$ (1)

So:

$$\int_C x\, ds = \int_0^1 x \sqrt{1 + (2x)^2}\, dx = \frac{1}{12}\left(1+4x^2\right)^{3/2}\Big|_0^1 = \frac{1}{12}\left(5\sqrt{5}-1\right) =$$

$$= 0.848$$

(1) (1)

(3) [2 Pts.] Set up the integral needed to compute the surface area of the parametric surface given by:

$$\mathbf{r}(u,v) = v^2\,\mathbf{i} - u\,v\,\mathbf{j} + u^2\,\mathbf{k}, \quad \text{for } 0 \leq u \leq 3,\ -3 \leq v \leq 3.$$

$\underline{r}_u = (0, -v, 2u)$

$\underline{r}_v = (2v, -u, 0)$ (1)

$\underline{r}_u \times \underline{r}_v = (-2u^2, 4uv, -2v^2)$

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{|\underline{r}_u \times \underline{r}_v|^2}\, dv\, du = \int_0^3 \int_{-3}^3 2\sqrt{u^4 + 4u^2v^2 + v^4}\, dv\, du$$

$\simeq 247.8$ [+1]

(2) [4 Pts.] Let $\mathbf{F}(x, y) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$ be a force field.

(a) Verify that $\mathbf{F}$ is a conservative vector field.

(b) Compute the work done by the force $\mathbf{F}$ to move an object along a smooth curve C joining $(0, 2, 0)$ to $(4, 0, 3)$.

(a) $\nabla \times \underline{F} = 0$. Since $\operatorname{dom} \underline{F} = \mathbb{R}^3$ and $\underline{F} \in C^1$, then $\underline{F}$ is conservative (1)

(b) Compute f s.t. $\underline{F} = \nabla f$

$$f_x = e^y \quad \Rightarrow \quad f = xe^y + C(y,z)$$

$$f_y = xe^y + C_y(y,z) = xe^y + e^z \quad \Rightarrow \quad C_y(y,z) = ye^z + C(z)$$

$$f_z = ye^z + C'(z) = ye^z \quad \Rightarrow \quad C'(z) = 0$$

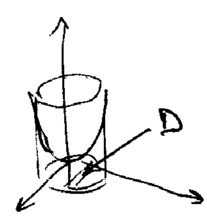
Thus $\boxed{f(x,y,z) = xe^y + ye^z}$ is a POTENTIAL FUNCTION (2)

$W = f(4,0,3) - f(0,2,0) = 4 - 2 = 2$ (1)

(4) [3 Pts.] Set up the surface integral:

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation. (You do not have to compute the integral, but you have to set up the double integral that solves the problem and specify the proper limits of integration)



$\underline{r}(x,y) = x\,\underline{i} + y\,\underline{j} + (x^2+y^2)\,\underline{k}$

$\underline{r}_x = \underline{i} + 2x\,\underline{k} \qquad r_y = \underline{j} + 2y\,\underline{k}$

(1) $\underline{r}_x \times \underline{r}_y = -2x\,\underline{i} - 2y\,\underline{j} + \underline{k}$ (upward orientation)

$\underline{F}(x,y,z) = P(x,y,z)\,\underline{i} + Q(x,y,z)\,\underline{j} + R(x,y,z)\,\underline{k} \qquad \underline{F}(\underline{r}(x,y)) = P(x,y)\,\underline{i} + Q(x,y)\,\underline{j} + R(x,y)\,\underline{k}$

Thus $\displaystyle\iint_S \underline{F} \cdot d\underline{S} = \iint_D \left[-2x\,P(x,y) - 2y\,Q(x,y) + R(x,y)\right] dA$ $\qquad D = \{(x,y) : x^2+y^2 \le 1\}$ (1)

(1)

$$\boxed{= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[-2x\,P(x,y) - 2y\,Q(x,y) + R(x,y)\right] dy\, dx}$$

See formula 10 at p. 467 $\qquad \displaystyle\iint_S \underline{F} \cdot d\underline{S} = \iint_D \underline{F} \cdot (\underline{r}_u \times \underline{r}_v)\, dA$