Math 242 Sec. 9 – Fall 2005 Name: SOLUTION A/B

FINAL EXAM -A

One page of notes, a hand calculator allowed ~~and your laptop are allowed~~. The test is divided in two parts: a multiple choice part and a traditional problem test part with possible partial credits. **Please, report the solution of the multiple choice part on the included solution page.** For the second part of the test, please, justify your answers and write clearly if you want credit for your work.

(1) [2 Pt.] Find the area of the parallelogram with vertices at $P = (1, 3, -2)$, $Q = (2, 1, 4)$ and $R = (-3, 1, 6)$.

(a) $A = 1$
(b) $A = 78$
(c) $A = \sqrt{1040}$
(d) $A = \sqrt{1124}$
(e) $A = \sqrt{1140}$
(f) $A = 82$
(g) $A = 62$
(h) $A = 0$

$\vec{PQ} = (1, -2, 6)$
$\vec{QR} = (-5, 0, 2)$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} i & j & k \\ 1 & -2 & 6 \\ -5 & 0 & 2 \end{vmatrix} = (-4, -32, -10$$

$|\vec{PQ} \times \vec{QR}| = \sqrt{1140}$

(2) [2 Pt.] Find the value(s) x such that the vectors $(3, 2, x)$ and $(2x, 4, x)$ are orthogonal.

(a) $x = -4$
(b) $x = -2$
(c) $x = 0$
(d) $x = 2$ or $x = 4$
(e) $x = -2$ or $x = 0$
(f) $x = -4$ or $x = -2$
(g) $x = -4$ or $x = 0$
(h) *never*

$6x + 8 + x^2 = 0$

$\Rightarrow x = -4$ o $x = -2$

(3) [2 Pts.] Find the equation of the plane containing the line: $\mathbf{r}(t) = (1 + t, 3 - 2t, -2 + 6t)$, and the point $R = (-3, 1, 6)$.

(a) $x - 8y + 3z = -29$

(b) $x + 8y + 3z = 19$

(c) $x - 8y = -22$

(d) $x + 8y = 25$

(e) $2x + 16y - 5z = 60$

(f) $-2x + 16y + 5z = 38$

(g) $\boxed{2x + 16y + 5z = 40}$

(h) $2x - 16y + 5z = -56$

$\underline{r}(0) = (1, 3, -2)$

$\underline{r}(1) = (2, 1, 4)$

$R = (-3, 1, 6)$

$2 + 16 \cdot 3 - 10 = 40$

$4 + 16 + 20 = 40$

$-6 + 16 + 30 = 40$

(4) [2 Pts.] Find the derivative $\frac{dy}{dx}$, where y is defined by

$$\cos(x-y) = x\,e^y.$$

(a) $\dfrac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$

(b) $-\dfrac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$

(c) $\dfrac{\sin(x-y)+e^y}{\sin(x-y)-e^y}$

(d) $-\dfrac{\sin(x-y)+e^y}{\sin(x-y)-e^y}$

(e) $\sin(x-y)-xe^y$

(f) $-\sin(x-y)-xe^y$

(g) $\sin(x-y)$

(h) $-\sin(x-y)$

$F(x,y) = \cos(x-y) - x e^y$

$F_x = -\sin(x-y) - e^y$

$F_y = +\sin(x-y) - x e^y$

$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = \dfrac{\sin(x-y)+e^y}{\sin(x-y)-xe^y}$

(5) [2 Pts.] Compute the following double integral (Hint: it may be useful to convert to polar coordinates):

$$I = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

(a) $I = \frac{1}{2}\pi$

(b) $I = \pi$

(c) $I = 18\,\pi$

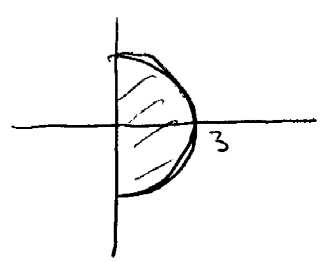
(d) $I = 9\,\pi$

(e) $I = \frac{9}{2}\pi$

(f) $I = \frac{81}{4}\pi$

(g) $I = \frac{81}{8}\pi$

(h) $I = \frac{27}{4}\pi$



$\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r\, r\, dr\, d\theta = \pi \left.\frac{r^3}{3}\right|_0^3 =$

$= 9\pi$

(6) [2 Pts.] Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 9$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

(a) $V = 27\sqrt{2}\,\pi$
(b) $V = 4\sqrt{2}\,\pi$
(c) $V = 9\sqrt{2}\,\pi/2$
(d) $V = 9\sqrt{2}\,\pi$
(e) $V = 8\sqrt{2}\,\pi/3$
(f) $V = 0$
(g) $V = 16\sqrt{2}\,\pi$
(h) $V = 3\pi^2/2$

$z = \sqrt{x^2+y^2} \iff \varphi = \frac{\pi}{4}$

$E = \{(\rho,\vartheta,\varphi) : 0 \le \rho \le 3,\ 0 \le \vartheta \le 2\pi,\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{2}\}$

$$V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 \rho^2 \sin\varphi \, d\rho \, d\vartheta \, d\varphi =$$

$$= 2\pi \left(-\cos\varphi\right)_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\rho^3}{3}\Big|_0^3 =$$

$$= \cancel{2}\pi \frac{\sqrt{2}}{\cancel{2}} 9$$

(7) [2 Pts.] Let $\mathbf{F}(x,y,z) = (4xz + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + 2(x^2 + z)\,\mathbf{k}$. Find the work W done by the vector field $\mathbf{F}$ in moving a particle along the curve C defined by $x(t) = \sin t, y(t) = \cos t - 1, z(t) = 2\cos t + 1$, for $0 \le t \le \pi/2$ (Hint: use the Fundamental Theorem for line integrals).

(a) $W = -6$
(b) $W = 0$
(c) $W = 13$
(d) $W = -9$
(e) $W = -5$
(f) $W = 4$
(g) $W = 5$
(h) $W = -4$

$f(x,y,z) = 2x^2z + xy^2 + z^2$

$\underline{r}(\frac{\pi}{2}) = (1,-1,1)$

$\underline{r}(0) = (0,0,3)$

$f(1,-1,1) = 2+1+1 = 4$

$f(0,0,3) = 9$

$W = 4 - 9 = -5$

(8) [2 Pts.] Compute the flux $\Phi = \iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^3\,\mathbf{i} + 2xz^2\,\mathbf{j} + 3y^2z\,\mathbf{k}$ and S is the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. (Hint: use Divergence Theorem).

(a) $\Phi = 16$
(b) $\Phi = 16\,\pi$
(c) $\Phi = 32\,\pi$
(d) $\Phi = \pi/2$
(e) $\Phi = 0$
(f) $\Phi = 8\,\pi$
(g) $\Phi = 64\,\pi$
(h) $\Phi = 5\pi/2$

$$\nabla \cdot \underline{F} = 3x^2 + 3y^2$$

$$\iiint_E (3x^2+3y^2)\, dV =$$

$$= \iint_D (4-r^2)\, 3r^2\, dA =$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\, 3r^3\, dr\, d\theta$$

$$= 2\pi \int_0^2 (12r^3 - 3r^5)\, dr$$

$$= 2\pi \left(12\frac{r^4}{4} - 3\frac{r^6}{6}\right)\Big|_0^2$$

$$= 2\pi \left(3\cdot 2^4 - \frac{2^6}{2}\right)$$

$$= 2\pi\, 2^4 (3-2) = \boxed{32\pi}$$

(9) [2 Pts.] Evaluate the surface integral $I = \iint_S xyz\, dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

(a) $I = 0$
(b) $I = \pi$
(c) $I = 3\pi/2$
(d) $I = \pi/2$
(e) $I = \pi/4$
(f) $I = 3\pi/4$
(g) $I = \pi/8$
(h) $I = \pi/16$

$\underline{r}(\varphi, \vartheta) = (\sin\varphi \cos\vartheta,\ \sin\varphi \sin\vartheta,\ \cos\varphi)$

$|r_\varphi \times r_\vartheta| = \sin\varphi$ $\qquad 0 \le \varphi \le \frac{\pi}{4}$

$0 \le \vartheta \le \frac{\pi}{2}$

$$S = \int_0^{2\pi} \int_0^{\pi/4} \sin^3\varphi \cos\varphi \cos\vartheta \sin\vartheta \, d\varphi \, d\vartheta = 0$$

Since $\int_0^{2\pi} \cos\vartheta \sin\vartheta = 0$

(10) [2 Pts.] Evaluate the following integral (hint: you might have to reverse the order of integration):

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx =$$

(a) $I = -\frac{1}{12}\cos(1)$
(b) $I = \frac{1}{12}\cos(1)$
(c) $I = \frac{1}{12}(1 - \cos(1))$
(d) $I = \frac{1}{12}(\cos(1) - 1)$
(e) $I = -\frac{1}{4}\cos(1)$
(f) $I = \frac{1}{4}\cos(1)$
(g) $I = \frac{1}{4}(1 - \cos(1))$
(h) $I = \frac{1}{4}(\cos(1) - 1)$

$$= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^1 \sin(y^3) \left.\frac{x^4}{4}\right|_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{4} y^2 \sin(y^3)\, dy = \frac{1}{12}\int_0^1 \sin u\, du$$

$u = y^3$

$du = 3y^2$

$$= \frac{1}{12}(-\cos u)\Big|_0^1$$

$$= \frac{1}{12}(1 - \cos(1))$$

(11) [2 Pts.] Find the parametric equation of the tangent line to the curve $\mathbf{r}(t) = (t^2, t^4, t^3)$ at the point point $P(1, 1, 1)$.

$$\left.\begin{aligned} \underline{r}'(t) &= (2t, 4t^3, 3t^2) \\ \underline{r}'(1) &= (2, 4, 3) \end{aligned}\right\} \quad (1\text{Pt})$$

Line
$$\begin{cases} x(t) = 1 + 2t \\ y(t) = 1 + 4t \\ z(t) = 1 + 3t \end{cases} \quad (1\text{Pt})$$

(12) [~~2~~ 3 Pts.] Set up ~~Evaluate~~ the line integral $\int_C x^{2/3}\, ds$ where C is the curve $\mathbf{r}(t) = (t^{3/2}, \cos(t), \sin(t))$ for $0 \le t \le 1$.

$$\underline{r}'(t) = \left(\frac{3}{2} t^{1/2}, -\sin t, \cos t\right) \quad (1\text{Pt})$$

$$|\underline{r}'(t)| = \sqrt{\frac{9}{4}t + \sin^2 t + \cos^2 t} = \sqrt{1 + \frac{9}{4}t} \quad (1\text{Pt})$$

$$\int_C x^{2/3}\, ds = \boxed{\int_0^1 t\sqrt{1 + \frac{9}{4}t}\, dt} \cancel{= \frac{4}{9}\int_1^{13/4} (u-1)\sqrt{u}\,\frac{4}{9}\,du} \quad (1\text{Pt})$$

$$\cancel{= \frac{16}{81}\int_1^{13/4} \left(u^{3/2} - u^{1/2}\right) du}$$

$$\cancel{u = 1 + \frac{9}{4}t}$$

$$\cancel{du = \frac{9}{4}\, dt}$$

$$\cancel{t = \frac{4}{9}(u-1)}$$

$=$

(13) [3 Pts.] Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = e^{-x}\,\mathbf{i} + e^{x}\,\mathbf{j} + e^{z}\,\mathbf{k}$$

and C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant and is oriented counterclockwise as viewed from above.

$$\nabla \times \underline{F} = e^{x}\,\underline{k}$$

S: plane $2x + y + 2z = 2$ or $z = 1 - x - \frac{1}{2}y$

for $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le 2 - 2x\}$

$$\oint \underline{F} \cdot d\underline{r} = \iint_S (e^{x}\underline{k}) \cdot d\underline{S} = \iint_D e^{x}\, dA \qquad (1\text{pt})$$

$$= \int_0^1 \int_0^{2-2x} e^{x}\, dy\, dx = \int_0^1 (2 - 2x)e^{x}\, dx \qquad (1\text{pt})$$

$$= \Big[(2 - 2x)e^{x} + 2e^{x}\Big]_0^1 = \boxed{2e - 4} \qquad (1\text{pt})$$