Math 715 – Spring 2005 Name:

FINAL EXAM

(1)[4 Pts] Let $f, g \in L^1_{loc}(\mathbb{R})$ and suppose that

$$\int f\,\phi = \int g\,\phi$$

for all $\phi \in C_c^\infty(\mathbb{R})$. Prove that $f = g$ a.e.

(2)[4 Pts] (a) Compute the (distributional) derivative of $\chi_{[a,b]}$.

(b) Compute the Fourier transform of the Heaviside function $H(x) = \chi_{(0,\infty)}$. (Hint: write $H(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(x)$).

(c) Compute the Fourier transform of the locally integrable function $\sin x$. (Hint: recall $\sin x = \frac{e^{2\pi i x} - e^{-2\pi i x}}{2i}$)

(3)[4 Pts] Suppose that $f(x)$ is a bandlimited function with band 1, that is, $\hat{f}(\xi) = 0$ for $|\xi| > 1/2$. Prove that

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\,\frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

This is known as the *Shannon sampling theorem.*

(Hint: write $\hat{f}(\xi)$ as a Fourier series in ξ. Then substitute this Fourier series into $f(x) = \int_{-1/2}^{1/2} \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi$.)