TEST #4

(1)[4 Pts] Let \( f, \hat{f} \in L^1(\mathbb{R}) \).
(a) Show that \( f \) is real and even if and only if \( \hat{f} \) is real and even. Show that in this case
\[
\hat{f}(\xi) = 2 \int_0^\infty f(x) \cos(2\pi x \xi) \, dx.
\]
(b) Show that \( f \) is real if and only if \( \hat{f}(\xi) = \overline{\hat{f}(-\xi)} \).

(2)[4 Pts] (a) Show that
\[
\left( \frac{t}{\pi(x^2 + t^2)} \right)^\wedge(\xi) = e^{-2\pi|\xi|t}.
\]
(Hint: compute \((e^{-2\pi|\xi|t})^\vee\)).
Use (a) to prove that
\[
\lim_{t \to 0} \int_{\mathbb{R}} e^{-2\pi|\xi|t} \hat{f}(\xi)e^{2\pi i \xi x} \, d\xi = f(x),
\]
with convergence in the \( L^1 \) norm.
(Hint: see the argument in the proof of Th.8.26)

(3)[4 Pts] The Hilbert transform is given by
\[
Tf = -\frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy,
\]
where \( f \in L^2(\mathbb{R}) \) and the integral is interpreted as
\[
\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x - y} \, dy.
\]
Show that \((Tf)^\wedge(\xi) = \text{sgn}(\xi) \hat{f}(\xi)\).
(Hint: write \( Tf = f * g_{\epsilon}(x) \), where \( g_{\epsilon}(y) = -\frac{1}{\pi i} y^{-1} \) for \( |y| > \epsilon \) (and 0 for \( |y| \leq \epsilon \)). Then take \( \epsilon \to 0 \) and use the fact that \( \lim_{R \to \infty} \int_0^R \frac{\sin x}{x} \, dx = \pi/2 \).