Math 715 – Spring 2005 Name:

TEST #1

(1)[3 Pts] Prove that if $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F)$.

(2)[3 Pts] The symmetric difference of sets $A, B \subset X$ is the set $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$. Let $(X, \mathcal{M}, \mu)$ be a finite measure space (i.e., $\mu(X) < \infty$ and, thus, $\mu(A) < \infty$ for each $A \subset X$). Show that $d(A, B) = \mu(A \triangle B)$ defines a semi-metric on $\mathcal{M}$, that is, $d(A, B) \geq 0$, $d(A, B) = d(B, A)$ and $d(A, C) \leq d(A, B) + d(B, C)$, for any $A, B, C \in \mathcal{M}$.

(3)[3 Pts] Prove that if $f : \mathbb{R} \mapsto \mathbb{R}$ is monotone, then f is Borel measurable.

(4)[3 Pts] Let f, g be measurable real-valued functions. The goal of this exercise is to show that $f + g$ is also measurable.

(a) Show that the condition " $\{x : f(x) > a\}$ is measurable for all $a \in \mathbb{R}$" holds iff it holds for all rational a.

(b) For rational a, prove that

$$\{x : f(x)+g(x) > a\} = \bigcup_{b \text{ rational}} \Big(\{x : f(x) > b\} \cap \{x : g(x) > a-b\}\Big).$$

(b) Conclude that $f + g$ is measurable, whenever f, g are.