FINAL EXAM

(1) An element $z \in H$ is in $\mathcal{R}(T)^{\perp}$ if and only if $\langle z, Tx \rangle = 0 \quad \forall x \in H$.

This is equivalent to $\langle T^{*}z, x \rangle = 0 \quad \forall x \in H$.

Thus $z \in \mathcal{R}(T)^{\perp} \iff T^{*}z = 0$.

This shows that $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{*}) \qquad (1)$

We will show that $\mathcal{R}(T)^{\perp\perp} = \mathrm{cl}\,\mathcal{R}(T). \qquad (2)$

In fact; ~~let N = R(T)~~. If $x \in \mathcal{R}(T)$, then $x \perp \mathcal{R}(T)^{\perp}$ and thus $x \in \mathcal{R}(T)^{\perp\perp}$.

This shows that $\mathcal{R}(T) \subset \mathcal{R}(T)^{\perp\perp}$.

Since $\mathcal{R}(T)^{\perp\perp} = (\mathcal{R}(T)^{\perp})^{\perp}$ is closed, ~~it~~ only remains to show that $\mathrm{cl}\,\mathcal{R}(T) \subset \mathcal{R}(T)^{\perp\perp}$.

Arguing by contradiction, let us assume that there is $z \in \mathcal{R}(T)^{\perp\perp}$ such that $z \perp \mathrm{cl}(\mathcal{R}(T))$. Then $z \in \mathcal{R}(T)^{\perp}$. But $z \in \mathcal{R}(T)^{\perp} \cap \mathcal{R}(T)^{\perp\perp}$ implies that $z = 0$. Thus $\mathrm{cl}(\mathcal{R}(T)) \subset \mathcal{R}(T)^{\perp\perp}$, and (2) holds.

Using (2) into (1) we conclude that $\mathrm{cl}\,\mathcal{R}(T) = (\mathcal{N}(T^{*}))^{\perp}$

(2) (a) Let $\mathcal{B}(H)$ be the space of bounded linear operators on H.

Consider the sequence of operators $\left(\sum_{k=0}^{n} T^k\right)$. For $m > n$, we have:

$$\left\| \left(\sum_{k=0}^{m} T^k\right) x - \left(\sum_{k=0}^{n} T^k\right) x \right\| = \left\| \left(\sum_{k=n+1}^{m} T^k\right) x \right\|$$

$$\le \sum_{k=n+1}^{m} \|T\|^k \|x\| \quad = \|x\| \frac{\|T\|^{n+1}}{1 - \|T\|}$$

Since $\|T\|^n \to 0$ as $n \to \infty$, it follows that $\left(\sum_{k=0}^{n} T^k\right)$ is a Cauchy sequence of operators in $\mathcal{B}(H)$.

Since $\mathcal{B}(H)$ is a Banach space, the $\sum_{k=0}^{n} T^k \to A \in \mathcal{B}(H)$.

Claim: $A = (I - T)^{-1}$.

For any $x \in H$, we have:

$$(I - T) \sum_{k=0}^{n} T^k x = (I - T)\left[I + T + T^2 + \dots + T^n\right] x$$

$$= (I - T^{n+1}) x$$

Thus $(I - T) A x = \lim_{n \to \infty} (I - T^{n+1}) x = x - \lim_{n \to \infty} T^{n+1} x$

But $\|T^{n+1} x\| \le \|T\|^{n+1} \|x\| \to 0$ as $n \to \infty$

Thus $(I - T) A = I$

Similarly $\left(\sum_{k=0}^{n} T^k\right)(I - T)\, x = \left[I + T + T^2 + \dots + T^n\right](I - T)\, x$

$$= (I - T^{n+1}) x$$

Thus $A(I - T) = I$.

This shows that $A = \sum_{n=0}^{\infty} T^k = (I - T)^{-1}$

(b) Let $A = I - T$

Then, by part (a) T is invertible since $\|A\| < 1$ and $(I - T) - I = -T$.

(3) (a) Let $\sum_n |\langle x, x_n\rangle|^2 = \|x\|^2 \quad \forall x \in H.$

Choose $x = x_j$, where x_j is a frame element. Then

$$\|x_j\|^2 = \sum_n |\langle x_j, x_n\rangle|^2 = |\langle x_j, x_j\rangle|^2 + \sum_{n\neq j} |\langle x_j, x_n\rangle|^2$$

$$= \|x_j\|^4 + \sum_{n\neq j} |\langle x_j, x_n\rangle|^2$$

Since $\|x_j\| = 1$, then it follows that

$$\sum_{n\neq j} |\langle x_j, x_n\rangle|^2 = 0 \qquad \text{and} \qquad \langle x_j, x_n\rangle = 0 \quad \forall n \neq j$$

This shows that (x_n) is an ORTHONORMAL SEQUENCE.

In addition, since $\sum |\langle x, x_n\rangle|^2 = \|x\|^2 \ \forall x \in H$, then (x_n) is an ONB.

(b) For $x \in H$, consider the operator S_N defined by

$$S_N x = \sum_{n \leq N} \langle x, x_n\rangle x_n .$$

It is clear that S_N is linear. To show boundedness, observe:

$$|\langle S_N x, y\rangle| \leq \sum_{n\leq N} |\langle x, x_n\rangle \langle x_n, y\rangle| \leq \left(\sum_n |\langle x, x_n\rangle|^2\right)^{1/2} \left(\sum_n |\langle y, x_n\rangle|^2\right)^{1/2}$$

$$\leq B \|x\| \|y\|$$

Thus $\quad \|S_N x\| = \sup_{\|y\|=1} |\langle S_N x, y\rangle| \leq B\|x\|$

and $\quad \|S_N\| = \sup_{\|x\|=1} \|S_N x\| \leq B$. Thus S_N is bounded

Observe: for $M > N$:

$$\|S_N - S_M\| = \sup_{\|x\|=1} \|S_N x - S_M x\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |\langle S_N x - S_M x, y\rangle| =$$

$$= \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left|\sum_{n=N+1}^{M} \langle x, x_n\rangle \langle x_n, y\rangle\right| \leq \sup_{\substack{\|x\|=1 \\ \|y\|=1}} \left(\sum_{n=N+1}^{M} |\langle x, x_n\rangle|^2\right)^{1/2} \left(\sum_{n=N+1}^{M} |\langle y, x_n\rangle|^2\right)^{1/2}$$

$$\to 0 \quad \text{as } N, M \to \infty$$

Thus S_N is Cauchy, and, by the completeness of $\mathcal{B}(H)$, then $S_N \to S \in \mathcal{B}(H)$.

Thus $Sx = \sum_{n=1}^{\infty} \langle x, x_n\rangle x_n$ is a bounded operator.

Also:
$$\langle Sx, y\rangle = \sum_n \langle x, x_n\rangle \langle x_n, y\rangle = \langle x, \sum_n \langle y, x_n\rangle x_n\rangle = \langle x, Sy\rangle$$

Thus $\quad S = S^* \qquad S$ is SELF-ADJOINT

③ (b) Alternative proof of boundedness of S:

$$\|Sx\|^4 = |\langle Sx, Sx\rangle|^2 = \left|\sum_n \langle x, x_n\rangle \langle x_n, Sx\rangle\right|^2$$

$$\leq \sum_n |\langle x, x_n\rangle|^2 \sum_n |\langle x_n, Sx\rangle|^2 \leq B\|x\|^2 \, B\|Sx\|^2$$

Thus $\|Sx\|^4 \leq B^2 \|x\|^2$

and $\|Sx\| \leq B\|x\|$.

(c) We have: $\langle Sx, x\rangle = \sum |\langle x, x_n\rangle|^2$

Thus $A\|x\|^2 \leq \langle Sx, x\rangle \leq \|Sx\| \, \|x\|$

Thus $\|Sx\| \geq A\|x\| \quad \forall x$

This implies that $\|Sx\| = 0 \iff \|x\| = 0$, and, thus, S is ONE TO ONE

S is invertible on its RANGE. Need to show that $R(S) = H$.

claim: $R(S)$ is closed.

let $f_n \in R(S)$, and $\|f_n - f_m\| \to 0$ as $n, m \to \infty$.

$f_n = Sx_n$ for some $x_n \in H$.

We have: $\|x_n - x_m\|^2 \leq \frac{1}{A} \langle S(x_n - x_m), x_n - x_m\rangle$

$$\leq \frac{1}{A} \|S(x_n - x_m)\| \, \|x_n - x_m\|$$

$$= \frac{1}{A} \|f_n - f_m\| \, \|x_n - x_m\| \to 0 \quad \text{as } n, m \to \infty$$

Thus $(f_n) = (Sx_n)$ is Cauchy and $f_n \to f = Sx$

Thus $R(S)$ is closed.

claim: $R(S)$ is dense in H.

Arguing by contradiction, suppose there is $f \in (R(S))^{\perp}$. Then

$\langle f, Sx\rangle = 0 \quad \forall x \in H$.

but $0 = \langle Sf, f\rangle \geq A\|f\|^2$ thus $f = 0$, and $R(S)$ is dense in H

We have shown that $R(S)$ is dense in H + closed.

Thus $R(S) = H$.

This concludes the proof that S is invertible on H.