Math 515 – Spring 2006 Name:

Final Exam

Solve the following problems.

(1) [4 Pts] Let T be a bounded linear operator on a Hilbert space H. Let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ be the null space and range of T, respectively. Show that

$$\mathrm{cl}(\mathcal{R}(T)) = (\mathcal{N}(T^*))^{\perp}.$$

(Hint: Show first that $(\mathcal{R}(T))^{\perp} = \mathcal{N}(T^*)$. Next show that $(\mathcal{R}(T))^{\perp\perp} = \mathrm{cl}(\mathcal{R}(T))$.)

(2) [5 Pts] (a) Prove that if T is a bounded linear operator on a Hilbert space H and $\|T\| < 1$, then $T - I$ is invertible (I is the identity operator) and

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

with convergence in the operator norm (notice that in the series $T^0 = I$).

(b) Use part (a) to deduce that if if T is a bounded linear operator on a Hilbert space H and $\|I - T\| < 1$, then T is invertible.

(3) [6 Pts] A sequence (x_n) in a Hilbert space H is a *frame* of H if there exist real constants $A, B > 0$ such that, for all $x \in H$:

$$A\,\|x\|^2 \leq \sum_{n=0}^{\infty} |\langle x, x_n\rangle|^2 \leq B\,\|x\|^2.$$

A and B are called the *frame constants.*

(a) Prove that if (x_n) is a frame of H with frame constants $A = B = 1$, and $\|x_n\| = 1$ for all n, then (x_n) is an orthonormal basis of H.

(b) Prove that if (x_n) is a frame of H, then the operator S, defined by $Sx = \sum_{n=0}^{\infty} \langle x, x_n\rangle\, x_n$, where $x \in H$, is a self-adjoint bounded linear operator on H.

(c) Prove that the operator S is invertible on H. (Hint: you have to show that S is one to one, and that the range $\mathcal{R}(S) = H$).