**Math 515 – Spring 2006** Name:

# TEST #4

*Solve the following problems.*

(1) [5 Pts] Let $A$ be a bounded linear operator on a Hilbert space $H$ and $S \subset H$ be a closed subspace. $S$ is an *invariant* subspace of $A$ if $Ax \in S$ for all $x \in S$.

(a) Prove that if $A$ is self-adjoint and $S$ is an invariant subspace of $A$, then $S^{\perp}$ is also an invariant subspace of $A$.

(b) Show that the if $S$ is an invariant subspace of $A$ and $A$ is not self-adjoint, then $S^{\perp}$ is not necessarily an invariant subspace of $A$ (i.e., find a counterexample).

(2) [5 Pts] (a) Let $P$ be a self-adjoint bounded linear operator on a Hilbert space $H$ such that $P^2 = PP = P$. Show that $P(H)$ is a closed subspace of $H$ and that any $x \in H$ has a unique decomposition

$$x = Px + z, \tag{1}$$

where $z = x - Px \in (P(H))^{\perp}$.

(b) Suppose that $P \neq 0$ satisfies (1), where $Px \in M$, $M$ is a closed subspace of a Hilbert space $H$ and $z \in M^{\perp}$. Show that $P$ is a linear self-adjoint operator and satisfies $P^2 = P$.

(3) [5 Pts] The range of a bounded linear operator on a Hilbert space $H$ need not be closed. Consider the multiplication operator $M(x(t)) = t\,x(t)$ on $L^2((0,1))$ and show that the range of $M$ is dense in $L^2([0,1])$, but is not closed.