SOLUTIONS QUIZ #3

(1) (a) $$Px = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$

LINEARITY
$$P(\alpha x + \beta y) = \sum_{i=1}^{n} \langle \alpha x + \beta y, x_i \rangle x_i$$
$$= \alpha \sum_{i=1}^{n} \langle x, x_i \rangle x_i + \beta \sum_{i=1}^{n} \langle y, x_i \rangle x_i$$
$$= \alpha Px + \beta Py$$

$P^2 = P$ Observe that, for $x_k \in \{x_i\}_{i=1}^{n}$, we have
$$Px_k = \sum_{i=1}^{n} \langle x_k, x_i \rangle x_i = \sum_{i=1}^{n} \delta_{ik} x_i = x_k$$

Thus: $$P(Px) = P \sum_{i=1}^{n} \langle x, x_i \rangle x_i = \sum_{i=1}^{n} \langle x, x_i \rangle Px_i = \sum_{i=1}^{n} \langle x, x_i \rangle x_i$$
$$= Px \quad \forall$$

(b) Let $z \in \mathcal{N}(P)$. That is $Pz = 0$.
Let $x \in \mathcal{R}(P)$. That is $Px = x$.

$$\langle x, z \rangle = \langle Px, z \rangle = \left\langle \sum_{i=1}^{n} \langle x, x_i \rangle x_i, z \right\rangle = \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, z \rangle$$
$$= \left\langle x, \sum_{i=1}^{n} \langle x_i, x \rangle z \right\rangle = \langle x, Pz \rangle = \langle x, 0 \rangle = 0$$

Thus $\mathcal{N}(P) \perp \mathcal{R}(P)$.

(2) We will use the THEOREM: An ORTHONORMAL SEQUENCE $\{x_n\}$ is COMPLETE IFF $\langle x, x_n \rangle = 0 \ \forall n$ implies $x = 0$.

Denote $C_n(t) = \cos 2\pi n t$, $S_n(t) = \sin 2\pi n t$. Recall that $\varphi_n(t) = C_n(t) + i S_n(t)$

Observe that, for $n = 0, 1, 2, \dots$ we have:

$$\langle f, \varphi_n \rangle = \langle f, C_n \rangle + i \langle f, S_n \rangle$$

$$\langle f, \varphi_{-n} \rangle = \langle f, C_n \rangle + i \langle f, S_{-n} \rangle = \langle f, C_n \rangle - i \langle f, S_n \rangle$$

Thus ~~implies that~~ $\langle f, C_n \rangle = 0$, $\langle f, S_n \rangle = 0 \ \forall n \in \mathbb{N}$ implies that $\langle f, \varphi_n \rangle = 0 \quad \forall n \in \mathbb{Z}$.

Since $\{\varphi_n\}$ is a COMPLETE ON sequence, this implies that $f = 0$. Thus $\{C_n, S_n : n \in \mathbb{N}\}$ is COMPLETE.

(3) Suppose that $\{y_n\}$ is NOT COMPLETE.

Then, there is an $x \in H$ such that $\langle x, y_n \rangle = 0 \ \forall n$.

We can write $\sum_n |\langle x, x_n - y_n \rangle|^2 = \sum_n |\langle x, x_n \rangle - \langle x, y_n \rangle|^2 = \sum_n |\langle x, x_n \rangle|^2 = \|x\|^2$ (1

(since $\{x_n\}$ is COMPLETE and $\langle x, y_n \rangle = 0 \ \forall n$)

On the other hand, $\sum_n |\langle x, x_n - y_n \rangle|^2 \leq \sum_n \|x\|^2 \|x_n - y_n\|^2 = \|x\|^2 \sum_n \|x_n - y_n\|^2$ (2

Combining (1) and (2) we have:

$$\|x\|^2 \leq \|x\|^2 \sum_n \|x_n - y_n\|^2$$

$$\sum_n \|x_n - y_n\|^2 \geq 1$$

This contradicts our assumption.

Thus it must be $\langle x, y_n \rangle = 0 \ \forall n$ implies $x = 0$.

That is, $\{y_n\}$ is COMPLETE.