Math 515 – Spring 2006 Name:

TEST #3

Solve 2 out of 3 problems. You will receive 3 extra-credit points on the third problem but no partial credits on this problem.

(1) [6 Pts] Let H be an Hilbert space. An operator $P : H \to H$ is a *projection* if: (a) P is linear and (b) $P^2 = P$, that is, $P(Px) = Px$, for all $x \in H$.

(a) Let $\{x_1, x_2, \dots, x_n\}$ be an orthonormal sequence in H. Show that

$$Px = \sum_{i=1}^{n} \langle x, x_i \rangle \, x_i$$

is a projection.

(b) A projection P is an *orthogonal projection* if the null space $\mathcal{N}(P)$ and the range space $\mathcal{R}(P)$ are orthogonal, that is $\langle x, z \rangle = 0$ for each $x \in \mathcal{R}(P)$, $z \in \mathcal{N}(P)$. Show that P, given above, is an orthogonal projection.

(2) [5 Pts] Show that the sequence of functions

$$\{1, \sqrt{2}\, \cos 2\pi x, \sqrt{2}\, \sin 2\pi x, \sqrt{2}\, \cos 4\pi x, \sqrt{2}\, \sin 4\pi x, \dots\}$$

is a complete orthonormal sequence in $L^2([-\frac{1}{2}, \frac{1}{2}])$. You do not have to show the orthogonality, only the completeness. To show this, use the fact that $\{\phi_n(x) = e^{2\pi i n x}\}_{n=-\infty}^{\infty}$ is a complete orthonormal sequence in $L^2([-\frac{1}{2}, \frac{1}{2}])$.

(3) [5 Pts] Prove that if $\{x_n\}_{n=1}^{\infty}$ is a complete orthonormal sequence in a Hilbert space H and $\{y_n\}_{n=1}^{\infty}$ is another orthonormal sequence satisfying

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1,$$

then $\{y_n\}_{n=1}^{\infty}$ is also complete.