TEST #2 SOLUTION

(1)(a) It is clear that $C^1[0,1]$ is a VECTOR SPACE

To show that $\|x\| = \max|x(t)| + \max|x'(t)|$ is a NORM, observe that:

$$\|x\| \geq 0$$

$\|x\| = 0$ implies that $x(t) = 0 \quad \forall t$

$$\|\lambda x\| = |\lambda|\,\|x\|$$

$$\|x+y\| = \max|x(t)+y(t)| + \max|x'(t)+y'(t)| \leq$$

$$\leq \max|x(t)| + \max|y(t)| + \max|x'(t)| + \max|y'(t)| = \|x\| + \|y\|$$

(b) Let $(x_n(t)) \subset C^1[0,1]$ be a Cauchy sequence.

Thus, given $\varepsilon > 0$, $\exists\, N(\varepsilon)$ s.t. $\|x_n(t) - x_m(t)\| < \varepsilon$ for $n,m > N(\varepsilon)$.

Since $x_n(t)$ is continuous on a closed interval, then $x_n(t)$ is uniformly continuous and $x_n(t) \to x(t)$ when $x(t)$ is continuous on $[0,1]$.

Similarly $x_n'(t)$ is uniformly continuous and $x_n'(t) \to y(t)$, when $y(t)$ is continuous on $[0,1]$. To complete the proof, we have to show that $y(t) = x'(t)$.

Since $x_n(t) \in C^1[0,1]$, then $x_n(t) = x_n(0) + \int_0^t x_n'(\tau)\,d\tau$

In the limit, $x(t) = \lim_{n\to\infty} x_n(t) = x(0) + \lim_{n\to\infty} \int_0^t x_n'(\tau)\,d\tau =$

$$= x(0) + \lim_{n\to\infty} \int_0^t y(\tau)\,d\tau$$

This implies that $x'(t) = y'(t)$.

(2) Since $f \in C^1[a,b]$, $f(a) < 0$, $f(b) > 0$ and $f'(x) > 0$ on $[a,b]$, then f has exactly ONE zero on $[a,b]$.

IF $Tx = x - \lambda f(x)$ is a contraction, then

$x_{n+1} = x_n - \lambda f(x_n)$ converges to the zero.

Need to ensure that T is a contraction.

$$Tx - Ty = x - y - \lambda\left(f(x) - f(y)\right).$$

By the MEAN VALUE THEOREM, there is a $\xi \in [a,b]$ s.t. $f(x) - f(y) = f'(\xi)(x-y)$

Then $|Tx - Ty| = |(x-y)(1 - \lambda f'(\xi))| = |1 - \lambda f'(\xi)|\,|x-y|$

T is a contraction if $|1 - \lambda f'(\xi)| < 1$ OR $-1 < 1 - \lambda f'(\xi) < 1$

This is satisfied when $\lambda > 0$

and $\lambda < \frac{2}{K_2}$

(3) We have

$$\|x+\lambda y\|^2 = \|x\|^2 + |\lambda|^2 \|y\|^2 + 2\,\mathrm{Re}(\bar{\lambda}\langle x,y\rangle) \qquad (1)$$

($\Rightarrow$) If $\langle x,y\rangle = 0$, then

$$\|x+\lambda y\|^2 = \|x\|^2 + |\lambda|^2\|y\|^2$$

and $\|x\|^2 \le \|x+\lambda y\|^2 \quad \forall\, \lambda \in \mathbb{C}$

($\Leftarrow$) Assume that $\|x+\lambda y\|^2 \ge \|x\|^2 \quad \forall \lambda \in \mathbb{C}$

By (1), this is equivalent to

$$|\lambda|^2\|y\|^2 + 2\,\mathrm{Re}(\bar{\lambda}\langle x,y\rangle) \ge 0 \qquad \forall \lambda\in\mathbb{C} \qquad (2)$$

Set $\bar{\lambda} = -t\dfrac{\langle y,x\rangle}{|\langle x,y\rangle|}$, with $t \in \mathbb{R}$, $t \ge 0$

Substitution into (2) gives

$$t^2\|y\|^2 - 2\,\mathrm{Re}\left(t\frac{|\langle x,y\rangle|^2}{|\langle x,y\rangle|}\right) = t^2\|y\|^2 - 2t\,|\langle x,y\rangle| \ge 0$$

Thus $|\langle x,y\rangle| \le -\frac{t}{2}\|y\|^2 \qquad \forall\, t \ge 0$

Take $\lim_{t\to 0}$ on both side and you get

$$|\langle x,y\rangle| = 0$$

Alternative Approach -

Set $\lambda = -\dfrac{\langle x,y\rangle}{\langle y,y\rangle}$

From (2):

$$\frac{|\langle x,y\rangle|^2}{\|y\|^2} - 2\,\mathrm{Re}\left\{\frac{|\langle x,y\rangle|^2}{\|y\|^2}\right\} \ge 0$$

$\Rightarrow \quad -|\langle x,y\rangle|^2 \ge 0 \quad \Rightarrow \quad |\langle x,y\rangle| \le 0$

$\Rightarrow \langle x,y\rangle = 0$