

TEST #2

(1) [4 Pts] $C^1[0, 1]$ is the space of continuous functions on $[0, 1]$, whose derivative is continuous on $(0, 1)$. Show that:

- (a) $C^1[0, 1]$ is a normed space, with norm $\|x\| = \max_{t \in [0, 1]} |x(t)| + \max_{t \in [0, 1]} |x'(t)|$;
- (b) $C^1[0, 1]$ is complete (and thus, it is a Banach space).

(2) [4 Pts] Using the Banach fixed point theorem, set up an iteration solving $f(x) = 0$, where $f \in C^1[a, b]$, $f(a) < 0$, $f(b) > 0$, and $0 < k_1 \leq f'(x) \leq k_2$, for all $x \in [a, b]$. Hint: use $Tx = x - \lambda f(x)$, for a suitable λ .

(3) [4 Pts] Let H be a Hilbert space, and $x, y \in H$. Prove that $x \perp y$ if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{C}$.