Unimodular Binary Hierarchical Models

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March 2, 2015
Example

- Let $T$ be the following $3 \times 2 \times 2$ table

<table>
<thead>
<tr>
<th></th>
<th>front</th>
<th>back</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- If we sum entries going down, we get the 2-way margin below. If we sum entries going left and back, we get the 1-way margin below.

$$
\begin{pmatrix}
3 & 6 \\
6 & 2
\end{pmatrix} \quad \begin{pmatrix}
5 \\
6
\end{pmatrix}
$$

- We are interested in the matrix that maps tables to margins
**Main Definition**

- $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is an integer vector, $d_i \geq 2$
- $\mathcal{C}$ denotes a simplicial complex on $[n]$
- $\text{facet}(\mathcal{C})$ denotes the inclusion-maximal faces of $\mathcal{C}$

**Definition**

Let $A_{\mathcal{C}, \mathbf{d}}$ be the matrix defined as follows:

- Columns are indexed by elements of $\bigoplus_{i=1}^{n} [d_i]$
- Rows are indexed by $\bigoplus_{F \in \text{facet}(\mathcal{C})} \bigoplus_{j \in F} [d_j]$
- Entry in row $(F, (j_1, \ldots, j_k))$ and column $(i_1, \ldots, i_n)$ is 1 if $i|_F = (j_1, \ldots, j_k)$
- All other entries are 0
Example

- Let $n = 3$ with $d_1 = 3$, $d_2 = 2$, $d_3 = 2$
- Let $C$ be the complex \[egin{array}{ccc}
1 & 2 & 3 \\
\end{array}\]
- Then $A_{C,d}$ is the following matrix:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
\{1\}, 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1\}, 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\{1\}, 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\{2, 3\}, 11 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\{2, 3\}, 12 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\{2, 3\}, 21 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\{2, 3\}, 22 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Definition

Let $A \in \mathbb{Z}^{n \times d}$ be an integral matrix with rank $n$. We say that $A$ is unimodular if for every $b \in \mathbb{NA}$, the following polyhedron has integral vertices

$$P_{A,b} := \{ x \in \mathbb{R}^d : Ax = b, x \geq 0 \}.$$
Motivating Question

Question
Given a simplicial complex $\mathcal{C}$ on $[n]$ and an integer vector $\mathbf{d} = (d_1, \ldots, d_n)$ with $d_i \geq 2$, is $\mathcal{A}_{\mathcal{C}, \mathbf{d}}$ unimodular?

Proposition
If $\mathcal{A}_{\mathcal{C}, \mathbf{d}}$ is unimodular, then for all $\mathbf{d}'$ with $\mathbf{d}' \leq \mathbf{d}$ componentwise, $\mathcal{A}_{\mathcal{C}, \mathbf{d}'}$ is also unimodular.

Therefore we restrict our attention to the binary case; i.e. where $d_1 = d_2 = \cdots = d_n = 2$. 
Motivating Question

Definition

\[ \mathcal{A}_C := \mathcal{A}_{C,(2,\ldots,2)}. \]

Question

Given a simplicial complex \( \mathcal{C} \) on \([n]\), is \( \mathcal{A}_C \) unimodular?
Definition (Cone Vertices)

If $\mathcal{C}$ is a simplicial complex on $[n]$, define $\text{cone}(\mathcal{C})$ to be the complex on $[n + 1]$ with facets

$$\text{facet}(\text{cone}(\mathcal{C})) = \{ F \cup \{n + 1\} : F \in \text{facet}(\mathcal{C}) \}.$$ 

We say that $n + 1$ a cone vertex.
Definition (Ghost Vertices)

If \( C \) is a simplicial complex on \( [n] \), define \( G(C) \) to be the simplicial complex on \( [n + 1] \) that has exactly the same faces as \( C \). We say that \( n + 1 \) is a ghost vertex.
Unimodularity-Preserving Operations

Definition (Alexander Duality)

If $C$ is a simplicial complex on $[n]$, then the *Alexander dual* complex $C^*$ is the simplicial complex on $[n]$ with facets

$$\text{facet}(C^*) = \{ [n] \setminus S : S \text{ is a minimal non-face of } C \}.$$
Definition

We say that a simplicial complex $C$ is *nuclear* if it satisfies one of the following:

1. $C = \Delta_k$ for some $k \geq -2$ (i.e. a simplex)
2. $C = \Delta_m \sqcup \Delta_n$ (i.e. a disjoint union of simplices)
3. $C = \text{cone}(D)$ where $D$ is nuclear
4. $C = G(D)$ where $D$ is nuclear
5. $C$ is the Alexander dual of a nuclear complex.

Theorem (B-Sullivant 2015)

*The matrix $A_C$ is unimodular if and only if $C$ is nuclear.*
Let $C$ be a simplicial complex on $[n]$.

**Definition (Deletion)**

Let $v \in [n]$ be a vertex of $C$. Then $C \setminus v$ denotes the induced simplicial complex on $[n] \setminus \{v\}$. 

![Diagram showing simplicial complexes with and without a vertex](image-url)
Definition (Link)
Let \( v \in [n] \) be a vertex of \( C \). Then \( \text{link}_v(C) \) denotes the simplicial complex on \( [n] \setminus \{v\} \) with the following facets:

\[
\text{facet}(\text{link}_v(C)) = \{ F \setminus \{v\} : F \text{ is a facet of } C \text{ with } v \in F \}.
\]

Definition (Simplicial Complex Minor)
Let \( C, D \) be simplicial complexes. If \( D \) can be obtained from \( C \) by taking links of vertices and deleting vertices, then we say that \( D \) is a minor of \( C \).
Theorem (B-Sullivant 2015)

The matrix $A_C$ is unimodular if and only if $C$ has no simplicial complex minors isomorphic to any of the following:

- $\partial \Delta_k \sqcup \{v\}$, the disjoint union of the boundary of a simplex and an isolated vertex
- $O_6$, the boundary complex of an octahedron, or its Alexander dual $O_6^*$
- The four simplicial complexes shown below

![Diagram of four simplicial complexes](attachment:image.png)
Proposition

Let $A$ be a matrix of full row rank. If we can row reduce $A$ to get the matrix $[I_n|D]$, then $A$ is unimodular if and only if $D$ is totally unimodular.

Question

Can we use Seymour’s decomposition for regular matroids [4] to prove our results?
Question

Given a simplicial complex $\mathcal{C}$ on $[n]$ and an integer vector $\mathbf{d} = (d_1, \ldots, d_n)$ with $d_i \geq 2$, is $A_{\mathcal{C}, \mathbf{d}}$ unimodular?

Corollary (B-Sullivant 2015)

If $A_{\mathcal{C}, \mathbf{d}}$ is unimodular then $\mathcal{C}$ is nuclear.

Question

Let $\mathcal{C}$ and $\mathbf{d}$ be specified by the figure below. For which values of $p$ and $q$ is $A_{\mathcal{C}, \mathbf{d}}$ unimodular?

![Diagram of simplicial complex with vertices labeled p, q, 2, and 3.]}
References

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