Towards Improved Scalability in Smart Grid Modeling: Simplifying Generator Dynamics Analysis via Spectral Graph Sparsification

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***Abstract*—Smart-grid technologies focus on the complex interactions between different components of the electricity grid, together with the computing, control and communication functionalities that will bring together this future smarter infrastructure. Investigating these complex dynamic interactions is crucial for the efficiency and robustness of the emerging smart grid.**

In particular, it is one of the key elements for smart-grids to establish the dynamics among generators when a disturbance or fault appears on transmission and distribution systems. In small-signal stability analysis, eigenvalues from a system matrix construction is an important component of the generator dynamics. Such methods allow us certain mathematical tractability for generators, and help to reveal their stability or system mode.

However, the recent trends in electricity grids, accompanied by increases in size and more tangled interconnections, raise challenges to using this approach. In general, the complexity of obtaining eigenvalues increases with the grid size and we especially emphasize that a system matrix becomes fully dense under the existence of algebraic buses. This hampers a fast diagnosis of generators, and the model complexity is not acceptable for analyzing large-scale electricity grids through eigenvalues.

For this reason, we propose here a complexity reduction method for investigating generator dynamics, by utilizing a *spectral sparsifier*. This is based on finding a sparse counterpart which conserves the dense system matrix properties. We explain the reasoning for this sparsification and test the proposed method in several electricity grids. Our results show that the generator dynamics can be analyzed based on much simpler grid topologies, while staying within a controllable small error-bound.

I. Introduction

The emergence of green technologies, such as wind and solar energy, as commercially viable suppliers of electric power is leading to a rapid increase in the size and complexity of already large and highly interconnected power grids [1] due to their spatial distributed nature. Furthermore, the introduction of smart grid technologies to predict and intelligently respond to the behaviors and actions of all electric power users connected to it, makes the timely monitoring of the state of generators, transformers and transmission lines particularly critical for their success. The premise of such technologies is to enable the system to operate close to its control and operational limits. In particular, disturbances or faults (even small in size) in transmission and distribution systems affect the dynamics generators with adverse effects on smart grid power systems.

Small-signal stability corresponds to the ability of the power system to maintain synchronism when subjected to small disturbances. A disturbance is considered small if the equations that describe the resulting response of the system may be linearized for the purpose of analysis. The small-signal stability problem is usually associated with insufficient damping of system oscillations. For such analysis, it is appropriate to disregard the transmission network and machine stator transients.

	IEEE30	IEEE118	IEEE300	NYISO	WSCC
Average degree	2.73	3.03	2.73	4.47	2.67
Number of buses	30	118	300	2935	4941

TABLE I
THE TOPOLOGICAL CONNECTIONS OF EXISTING ELECTRICITY GRIDS [4]

The dynamics of the generators and other devices are represented by differential equations. The result is that the complete system model is described by a large set of ordinary differential and algebraic equations. Eigenvalue analysis is a prime tool to investigate the stability of the dynamics of the system [2], [3]. However, this approach proves challenging in the presence of very large systems.

As mentioned, an electric power system can be described by a set of differential (active) and algebraic (passive) equations. They express the generator dynamics and the network structure, respectively. In small-signal stability analysis, the algebraic equations are eliminated by reduction techniques and the system is reduced to differential equations only. This provides mathematical tractability to establish relations among generators. From the reduced differential equations, a *system matrix* $A_{\mathbf{sys}}$ is obtained by using a linearization at operating points. The eigenvalues of the matrix include information to describe the dynamics of generators and are used to indicate their stability or system modes.

In general, the complexity upper-bound for eigenvalues is $O(n!)$ and $O(n^3)$ under Laplace expansion and LU decomposition methods, respectively, where n is the number of generators. Therefore, the complexity to investigate generator dynamics also increases with the electricity grid size. In the literature, one can find several approaches for efficient eigenvalue computations in large-scale systems [3], [5], [6].

As shown in Section II-A, the system matrix involves a Laplacian sub-matrix $\mathbf{L}$ which represents the network topology. In Table. I, we provide topological information for several electricity grids. It can be seen that the average edge degree remains fairly small number (2$\sim$4) regardless of the network size (i.e., the number of buses). This implies that the system matrix will be more sparse in a large power grid. For sparse matrices, the complexity of calculating their eigenvalues depends on the number of nonzero elements and thus offers computational savings [7], [8], a potentially significant advantage when considering large systems.

Unfortunately, we can not directly take advantage of this facts in small signal stability analysis, since in the process of eliminating algebraic buses, the sub-matrix $\mathbf{L}$ does not preserve the physical topology information anymore. Moreover, its graph representation becomes complete (i.e., the matrix $\mathbf{L}$ is a fully dense matrix [9], [10]). Therefore the objective of this study is to discuss a method that leads to sparsify the corresponding dense Laplacian system matrix. The sparsifier utilizes results from *spectral graph theory*, supports the generator dynamics within an ϵ-bound error and the same eigenvalue order from small-signal stability analysis.

The remainder of this paper is organized as follows: In Section 2, we review the small-signal stability analysis and a reduced power system model. In Section 3, we discuss the concept of a spectral sparsifier. We tailor such a sparsifier in order to utilize it into electricity grids and propose an algorithm to analyze the dynamics of generators. For several electricity grid test-beds, we apply the proposed algorithm and compare the results with the generator dynamic from ones in traditional small-signal stability analysis in Section 4. Finally, we conclude this paper with summary and discussion.

II. Review of Small-Signal Stability Analysis

A. System Matrix $A_{\textsf{sys}}$ Construction and Eigenvalues

Consider a n machine (generator) power system. The dynamic model of each generator is given by the following swing equation:

$$\dot{\delta_i} = \Omega\omega_i,\ 2H_i\dot{\omega_i} + D_i\omega_i = P_{mi} - P_{ei},\ i = \{1, \ldots, n\} \quad (1)$$

where ω_i and δ_i are the rotor speed and angle, D_i is a damping coefficient, P_{mi} and P_{ei} are the mechanical power input and electrical power output of generator i. The constant Ω and H are the conversion factor from per-unit (p.u.) speed to rad/s and a machine inertia, respectively.

We denote by $\widetilde{V}$ and $\widetilde{V}^*$ the values for a voltage phasor and its complex conjugate, respectively, and Y^* implies values for conjugated line admittance. Then, the electrical power transferred from generator i into the neighboring bus set $\mathcal{N}_i$ is defined by $P_{ei} = \sum_{j\in\mathcal{N}_i} \mathbf{Re}(\widetilde{V}_i\widetilde{V}_j^* Y_{ij}^*)$, where $\mathbf{Re}(\cdot)$ denotes the real part. Under a lossless line condition, this becomes $P_{ei}{=}\sum_{j\in\mathcal{N}_i} |V_i||V_j|\mathbf{Im}(Y_{ij})\sin(\delta_i - \delta_j)$, where $\mathbf{Im}(\cdot)$ is the imaginary part [2].

We set $c_{ij} = |V_i||V_j|\mathbf{Im}(Y_{ij})$ and assume that the system consists of lossless lines with $\Omega = 1$. These allow us to rearrange (1) as follows:

$$\begin{aligned} \dot{\delta_i} &= \omega_i \\ \dot{\omega_i} &= \tfrac{1}{2H_i}(-\textstyle\sum_{j\in\mathcal{N}_i} c_{ij}\sin(\delta_i - \delta_j) + P_{mi} - D_i\omega_i) \end{aligned} \quad (2)$$

where $i \in \{1, \ldots, n\}$. The Jacobian matrix of (2) at the equilibrium points with a state set $[\delta_1, \ldots, \delta_n, \omega_1, \ldots, \omega_n]^T$ (i.e., linearization) leads to a *system matrix* $A_{\textsf{sys}}$ of size $2n \times 2n$:

$$A_{\textsf{sys}} = \mathbf{J}(\delta_1, \ldots, \delta_n, \omega_1, \ldots, \omega_n)|_{\delta_1=\delta_{10},\ldots,\omega_n=\omega_{n0}} \quad (3)$$

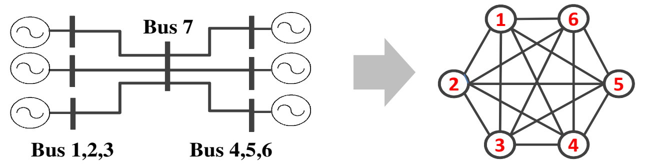


Fig. 1. 7 bus power grid with one algebraic node: Elimination of the algebraic bus 7 by using (6) induces a *complete* admittance graph among generators.

Under linearization, the stability and system mode of generators depend on the eigenvalues of this system matrix [11] and it has the following sub-matrix expression:

$$A_{\textsf{sys}} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{L} & -\text{diag}(\mathbf{D}) \end{bmatrix} \quad (4)$$

where $\mathbf{I}$ is the identity matrix, diag$(\mathbf{D})$ is a diagonal matrix of damping terms and $\mathbf{L}$ is a Laplacian matrix. When there is no algebraic bus, the sub-matrix $\mathbf{L}$ represents a physical electricity grid topology by the following definition:

$$\mathbf{L}_{ij} = \begin{cases} -\sum_{j,i\neq j}^{n} w_{ij} & \text{if } i = j \\ w_{ij} & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $w_{ij} = \frac{c_{ij}}{2H_i}\cos(\delta_{i0} - \delta_{j0})$. When damping terms are neglected (i.e., $\mathbf{D} = 0$), the eigenvalues of $A_{\textsf{sys}}$ are $\pm j\sqrt{|\lambda(\mathbf{L})|}$. This provides a computational tool for *system modes*, given by the imaginary part of the eigenvalues. Specifically, they describe the degree of generator oscillation under disturbances or faults. These values establish whether generators are independent or interact in some way; hence, they are useful in distinguishing intra- and inter-area generators.

The real part of the eigenvalues determines the stability of generators. We note that machine stability can not be determined when damping terms are ignored. In the Laplacian ($\mathbf{L}$) of a positive edge graph (e.g., an electricity grid with positive admittance lines), its eigenvalues are always real [12]. Therefore, all eigenvalues of the system matrix (i.e., $\pm j\sqrt{|\lambda(\mathbf{L})|}$) are purely imaginary and they can not indicate the stability of generators.

B. Reduced Power System Model

In electricity grids, there is a method to remove an algebraic (passive) bus under a constant load assumption, for the purpose of mathematical tractability of generators. We introduce *Kron reduction* based on Gaussian elimination [2], [3], [10].

In the system under consideration, we differentiate generator (active) buses and algebraic buses. We define $\mathcal{G}$ and $\mathcal{A}$ as their sets, respectively, where $|\mathcal{G}|{=}n$ and $|\mathcal{A}|{=}m$. Then, the size of the admittance matrix $\mathbf{Y}$ is $(n + m) \times (n + m)$ and the reduction of an algebraic bus k is performed by:

$$\mathbf{Y}_{ij}^{\text{New}} = \mathbf{Y}_{ij} - \mathbf{Y}_{ik}\mathbf{Y}_{kj}/\mathbf{Y}_{kk} \quad (6)$$

In this operation, the elements of the k^{th} row and column in $\mathbf{Y}$ become 0. Removing them results in a matrix $\mathbf{Y}^{\text{New}}$ with its size reduced by 1 and repeating this process m times induces an admittance matrix only involving generator buses.

When a system includes algebraic buses, the eigenvalue analysis shown in II-A becomes available by using the size-reduced admittance matrix. However, this reduced matrix is now a *full matrix* [9], [1] (i.e., $\mathbf{Y}_{ij}{\neq}0$ for $i, j{\in}\mathcal{G}$ as shown in

Fig. 1). As a result, the matrix density of $A_{\mathbf{sys}}$ also increases. In addition, the sub-matrix $\mathbf{L}$ also becomes full, since each element includes c_{ij} terms. As a result, the complexity of eigenvalue analysis becomes $\Omega(n!)$ or $\Omega(n^3)$, as discussed in the introductory section, thus particularly challenging for large scale power systems.

III. An Approach for a Scalable Stability and Mode Analysis via Graph Sparsification

We introduce next a spectral sparsifier that exhibits some desirable properties to address the challenge above.

A. Reduced Power System Model and Our Motivation

We consider a n generator power system with algebraic buses. Suppose that $\mathbf{Y}^R$ represents its reduced admittance matrix among generator buses, full with size $n \times n$. From (2), this system is described with $2n$ differential equations.

We start from the case when damping terms are ignored (i.e., a sub-matrix $\mathbf{D}=0$). Then, system modes can be obtained from $\mathbf{L}$, as $\lambda(A_{\mathbf{sys}})=\pm j\sqrt{|\lambda(\mathbf{L})|}$ and the following n differential equations are sufficient for the analysis of generator dynamics:

$$\ddot{\delta}_i = \frac{1}{2H_i}(P_{mi} - \sum_{j=1}^{n} c_{ij}\sin(\delta_i - \delta_j)), \quad i \in \{1,\dots,n\} \quad (7)$$

where $c_{ij} = |V_i||V_j|\mathbf{Im}(\mathbf{Y}^R_{ij})$.

After linearizing (7) at operating points $\delta_{i0},\dots,\delta_{n0}$, the Laplacian $\mathbf{L}$ in (4) becomes $\mathbf{J}(\delta_1,\dots\delta_n)$ given by:

$$\mathbf{L} = \begin{bmatrix} -\sum\limits_{j\neq 1}^{n}\mathbf{L}_{1j} & \dots & \frac{c_{1n}}{2H_1}\cos(\delta_{10}-\delta_{n0}) \\ \frac{c_{21}}{2H_2}\cos(\delta_{20}-\delta_{10}) & -\sum\limits_{j\neq 2}^{n}\mathbf{L}_{2j}\dots & \frac{c_{2n}}{2H_2}\cos(\delta_{20}-\delta_{n0}) \\ \vdots & \ddots & \vdots \\ \frac{c_{n1}}{2H_n}\cos(\delta_{n0}-\delta_{10}) & \dots & -\sum\limits_{j\neq n}^{n}\mathbf{L}_{nj} \end{bmatrix} \quad (8)$$

Each element value $\mathbf{L}_{ij}$ is determined by generator parameters (δ_{i0} and H_i), bus voltages (V_i and V_j) and the reduced admittance $\mathbf{Y}^R_{ij}$. Also, this matrix is in principal time-varying (i.e., $\mathbf{L}(t)$).

The eigenvalues of this matrix represent system modes by taking a square-root. However, its computational complexity rapidly increases with n, since $\mathbf{L}$ is a full $n \times n$ matrix. The goal is to turn it into a sparse matrix (i.e., $\mathbf{L}_{ij}=0$ for most elements) to facilitate analysis of large scale systems.

There is a physical meaning to the sparsification. In (7), the rotor angle variation $\ddot{\delta}$ of any generator is affected by all other $n-1$ generators and each element in (8) represents the correlation degree generators. Therefore, setting $\mathbf{L}_{ij}=0$ implies that we disregard correlations among generators below a certain threshold. Note that an eigenvalue analysis under this sparsification does not guarantee the correct dynamics for generators.

To overcome this issue, our approach is to offset the effect of sparsification (i.e., edge removals) by weight-controlling the remaining edges. This has similarities with finding a simplified graph structure, while maintaining its original properties as shown in Fig. 2.

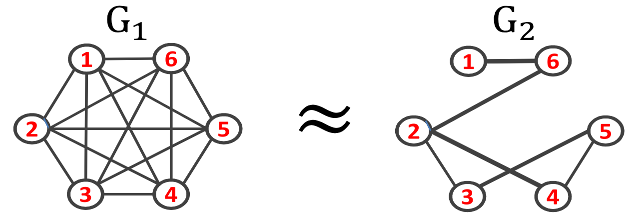


Fig. 2. The spectral property conservation: Eigenvalues of G_1 can be equal or only have a small error even under the simpler graph G_2.

B. Spectral Based Sparsifier

In graph theory, the purpose of spectral sparsifiers is to find a simplified structure of the original graph, while preserving key graph properties (see [7], [8], [12] and references therein). We consider next the construction efficiency for the simplified structure and a corresponding eigenvalue error-bound.

The error-bound can be attained through the comparison of graph Laplacians. When $\overline{\mathbf{L}}$ and $\mathbf{L}$ are Laplacian matrices for an original and a simplified graph, sparsifiers specifically focus on finding $\mathbf{L}$ which satisfies the following condition:

$$\forall x \in \mathcal{R}^n \quad (1-\epsilon)x^T\mathbf{L}x \leq x^T\overline{\mathbf{L}}x \leq (1+\epsilon)x^T\mathbf{L}x \quad (9)$$

From the Courant-Fisher Theorem [12], the eigenvalue λ_i is given by:

$$\lambda_i = \max_{S:dim(S)=k} \min_{x\in S} \frac{x^T\mathbf{L}x}{x^Tx} \quad (10)$$

Eigenvalues between $\mathbf{L}$ and $\overline{\mathbf{L}}$ have a one-to-one correspondence. Crucially, eqs. (9) and (10) indicate that each eigenvalue of $\mathbf{L}$ is within an error ϵ compared to the one of $\overline{\mathbf{L}}$. Also, the order of eigenvalues is conserved, as $\lambda_i \leq \lambda_{i+1}$ if $\bar{\lambda}_i \leq \overline{\lambda}_{i+1}$.

In [7], a nearly-linear time *random-sampling* algorithm is provided, which satisfies (9) by using an effective resistance (R_e) among existing sparsifiers. Suppose $G_o(V,E)$ and $G_s(V,E')$ are the original graph and its sparsified version, where $|E'| < |E|$ and $|V| = n$. The edge set E' comes by sampling edges from E. When q samples are to be selected and w_e denote the generic edge's weight, the algorithm is given by:

Algorithm 1 [7] $G_s = Sparsify(G_o, q)$

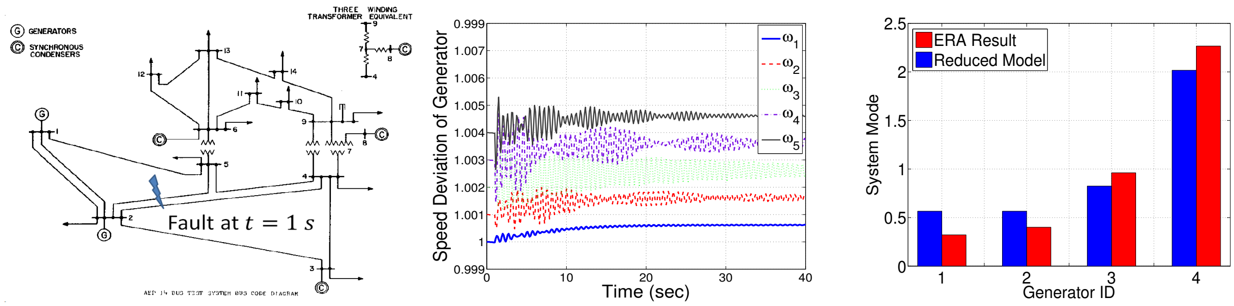
Choose a random edge e on G_o with probability p_e, where p_e is proportional to w_eR_e. Then, add the edge e to G_s with weight $\frac{w_e}{qp_e}$. Take q samples independently with replacement and sum their weights, if an edge is chosen more than once.

For the sparsified graph G_s, its edge set E' is determined by the number of samples q, where $|E'| \leq q$ from duplicated edge selections. This sparsifier limits the error bound to $|E| = O(n\log n/\epsilon^2)$ (i.e., a larger number of samplings generates a graph whose spectral properties are more close) [7].

The key of this algorithm is the use of effective resistance. [13], [14] show that the following relationship holds between a graph Laplacian $\mathbf{L}$ and an effective resistance R_e:

$$R_e(i,j) = \mathbf{L}^+_{ii} + \mathbf{L}^+_{jj} - \mathbf{L}^+_{ij} - \mathbf{L}^+_{ji} \quad (11)$$

where $\mathbf{L}^+$ is the pseudo-inverse of $\mathbf{L}$. Specifically, we see that R_e and $\mathbf{L}^+$ are proportional, since $R_e(i,j) = (\chi_i - \chi_j)^T\mathbf{L}^+(\chi_i - \chi_j)$, where χ_i is the elementary unit vector with



(a) IEEE 14 BUS Topology (b) Time-Domain Rotor Speed Variation (c) System Mode Comparison

Fig. 3. System mode comparisons between a PSAT Simulation and a reduced power system model when generator 5 is a reference in IEEE 14 BUS system

1 on position i. In addition, the inverse expression of (9) is given by:

$$\forall x \in \mathcal{R}^n \quad \frac{1}{(1+\epsilon)} x^T \mathbf{L}^+ x \le x^T \bar{\mathbf{L}}^+ x \le \frac{1}{(1-\epsilon)} x^T \mathbf{L}^+ x \tag{12}$$

Crucially, eqs. (11) and (12) indicate that random sampling proportional to R_e has the capability of bounding the eigenvalue error ϵ in that $R_e(i,j)$ is also proportional to $\mathbf{L}^+_{ij}$.

C. Application to Electricity Grids

A Laplacian matrix is always properly defined for any graph with positive edges and the random-sampling sparsifier provides its sparse counterpart. In a power system with algebraic buses, its sub-matrix (8) fulfills the Laplacian definition (5). Hence, analyzing system modes can be done effectively through an application of Alg. 1. Further, an ϵ control is provided for the analysis, due to the error being bounded by $|E| = O(n \log n/\epsilon^2)$.

However, this approach is not without challenges, including:

- **Time-varying matrix L:** The matrix $\mathbf{L}$ is time-varying (i.e., $\mathbf{L}(t)$), since machine parameters (δ_{i0}) and bus voltages (V_{i0}) change with time. Hence, the above method requires the use of a sparsifier at every eigenvalue analysis instant.
- **Sub-matrix sparsification in stability analysis**: For stability analysis, the sub-matrix $\mathbf{D}$ (i.e., damping coefficients) can not be neglected (see II-A). Thus, we need eigenvalues from a system matrix $A_{\mathbf{sys}}$, not from $\mathbf{L}$. In the view of $A_{\mathbf{sys}}$, the prior method is a sub-matrix sparsification. [15], [16] show that this sub-matrix sparsification can not generally preserve the spectral properties of $A_{\mathbf{sys}}$, except when all sub-matrices are Hermitian.

Instead of sparsifying the sub-matrix $\mathbf{L}$ which includes all dynamic parameters, we choose a grid topology sparsification to resolve the issues above. As shown in II-B, an original admittance matrix $\mathbf{Y}$ involves the physical topology of grids and its mathematically reduced matrix $\mathbf{Y}^R$ indicates a connection topology among generator buses. The random-sampling sparsifier can be used directly on the reduced matrix, since any admittance matrix also satisfies the Laplacian definition, given by (5).

By using this approach, we obtain the following advantages: First, the sparsified admittance matrix $\mathbf{Y}^R$ remains static if there is no structural loss. This relaxes the sparsification requirement for every eigenvalue analysis instant. Second, structure sparsification is independent from the submatrix sparsification problem, as $A_{\mathbf{sys}}$ itself becomes sparse autonomously.

Based on these points, we describe a scalable eigenvalue detection algorithm for large-scale electricity grids in Alg. 2. In the following section, we simulate the algorithm for several power grid test-beds and show its effectiveness.

Algorithm 2 Eigenvalue Detection for Large Power Grids

Require: $\mathcal{G}$ and $\mathcal{A}$ are sets for generator buses and algebraic buses, respectively. q is the number of samples.

1. $\mathbf{Y}^R \leftarrow$ Iterate (6) for Bus k, where $k \in \mathcal{A}$
2. $\mathbf{Y}^{sparse} = \textit{Sparsify}(\mathbf{Y}^R, q)$
3. $A_{\mathbf{sys}} \leftarrow$ Put $V_i, \delta_i, H_i, \mathbf{Y}^{sparse}_{ij}$ in (8), where $i, j \in \mathcal{G}$
4. Compute $\texttt{EIG}(A_{\mathbf{sys}})$

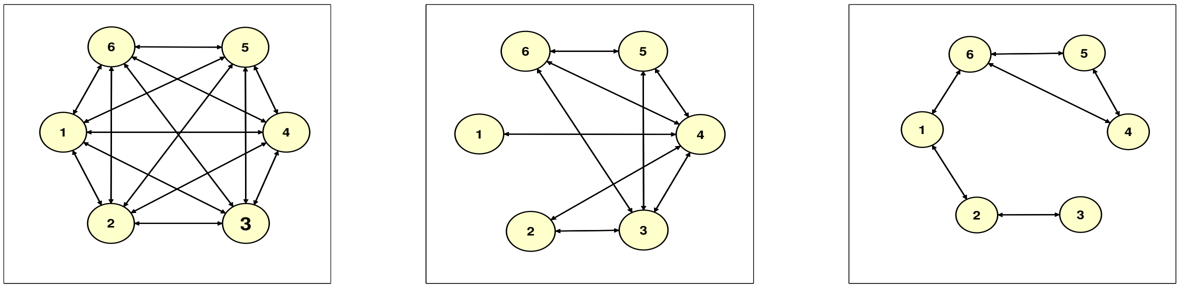
IV. Test Cases

We illustrate the effectiveness of our approach with data from power-system archives [17]. The suggested method sparsifies the admittance matrix, obtained by using the reduced power system model in II-B. Through comparisons between Power System Analysis Toolbox (PSAT) [18] simulations and analytical values, we show the validity of using a reduced power system model. Subsequently, we compare eigenvalues between an original matrix and a sparsified matrix attained through Alg. 2, while varying the sparsification degree.

A. Validation of Using a Reduced Power System Model

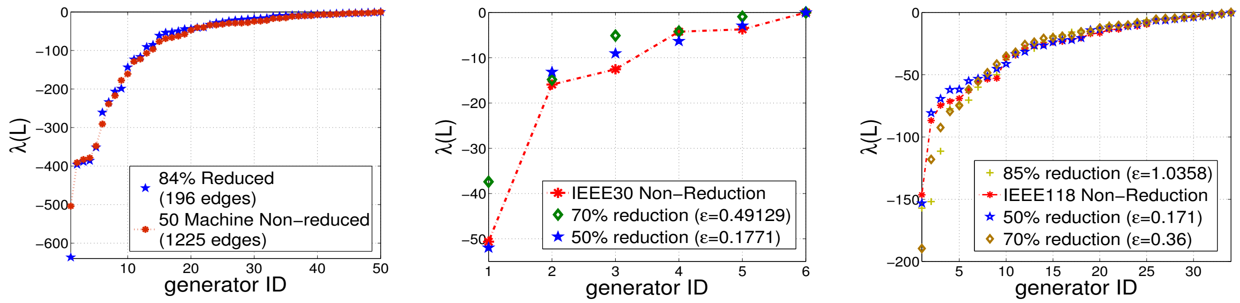
By utilizing PSAT and a IEEE 14 Bus system [17], we compare system modes between a simulated and a reduced power system model. Fig. 3(a) shows the topology of the IEEE 14 bus which has 5 generators and 30 lines. At $t = 1$ s, we set off a line fault between Buses 2 and 4 in order to see generator dynamics. Fig. 3(b) plots the speed deviation of generators during $t = (0, 40]$. We observe the relative dynamics of other generators by setting generator 5 as a reference. By using the Eigensystem Realization Algorithm (ERA) [19], we extracted eigenvalues for the relative model. We use 30 oscillatory frequencies with residues which imply the participation degree of time-response at a frequency.

For theoretical comparison, we build a Laplacian matrix $\mathbf{L}$ with states $[\delta_1, \ldots, \delta_5]^T$ by using a reduced admittance matrix and machine parameters in PSAT. The same relative model is



(a) Reduced Admittance $\mathbf{Y}_R$ (b) 40% sparsified ($q = 10$) (c) 60% sparsified ($q = 8$)

Fig. 4. A structural sparsification of IEEE 30 BUS: The reduced graph (b) and (c) preserve the spectral graph properties of (a) with less edges.



(a) 50 Machine System (b) IEEE 30 BUS (c) IEEE 118 BUS

Fig. 5. Eigenvalue comparison results and error-bound ϵ for several electricity grid test-beds after the sparsification of $\mathbf{L}$ by using Alg. 2.

considered (i.e., we set the state vector as $[\delta_1 - \delta_5, \ldots, \delta_4 - \delta_5]^T$) and we compute system modes by using $\pm j\sqrt{|\lambda(\mathbf{L})|}$. Fig. 3(c) shows the similarity of system modes between the reduced model and the ERA result.

B. Testing the Algorithms

For the IEEE 30, 118 Bus and a 50 machine system, we test the performance of Alg. 2. As system data in [17] do not include the inertia of generators H_i, we use a vector $H = [5.148, 6.54, 6.54, 5.06, 5.06]$ in PSAT for the IEEE 14 Bus model. In larger power systems, this vector is repeated for the inertia of generators.

1) Sparsification of the Admittance Matrix ($\boldsymbol{Y}_R$): In any power grid with passive buses, the reduced admittance matrix obtained via Kron reduction is expressed by a complete graph and our algorithm finds a simpler graph form with fewer edges. For the IEEE 30 Bus system which has 6 generators and 41 line connections, we show its sparsification results. In Fig. 4, (a) $\mathbf{Y}_R$ is complete with 6 nodes, while (b)-(c) are after 40% and 60% sparsification, respectively. Recall that the edge weights on a sparsified graph have been re-adjusted to conserve the spectral properties of an original graph. As an example, we show two adjacency matrices (eqs. (13) and (14)) for graphs in Fig. 4(a) and (c). Elements in the adjacency matrices represent edge weights.

$$adj(\mathbf{Y}_R) = \begin{pmatrix} 0 & 20.64 & 0.34 & 1.87 & 0.21 & 0.45 \\ 20.64 & 0 & 6.24 & 5.45 & 0.56 & 0.81 \\ 0.34 & 6.24 & 0 & 2.84 & 0.27 & 0.25 \\ 1.87 & 5.45 & 2.84 & 0 & 1.53 & 1.42 \\ 0.21 & 0.56 & 0.27 & 1.53 & 0 & 0.66 \\ 0.45 & 0.81 & 0.25 & 1.42 & 0.66 & 0 \end{pmatrix} \quad (13)$$

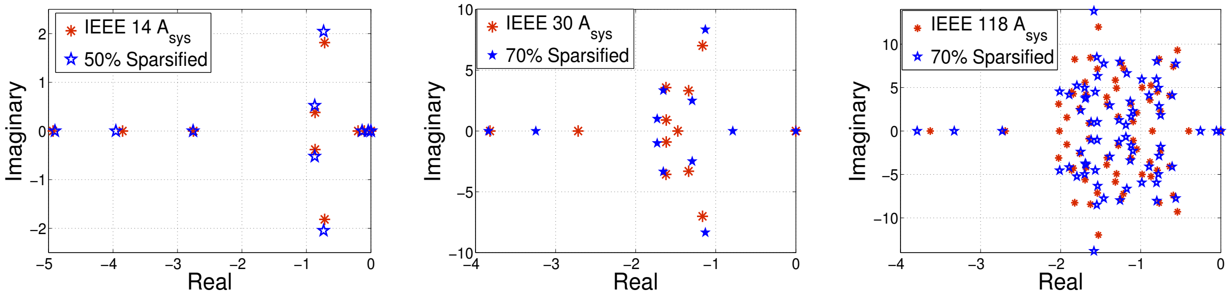
$$adj(\mathbf{Y}_R^{60\%}) = \begin{pmatrix} 0 & 43.09 & 0 & 0 & 0 & 1.84 \\ 43.09 & 0 & 5.63 & 0 & 0 & 0 \\ 0 & 5.63 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.84 & 1.99 \\ 0 & 0 & 0 & 1.84 & 0 & 1.32 \\ 1.84 & 0 & 0 & 1.99 & 1.32 & 0 \end{pmatrix} \quad (14)$$

2) System Mode Detection Tests from $\boldsymbol{L}$: For the 50 machine system which has 145 buses and 453 lines [17], we compute its reduced admittance matrix $\mathbf{Y}_R$. Due to its complete connections among all 50 generators, the number of edges is 1225. The edge number is reduced to 196 with new edge weight assignments through Alg. 2 (84% edge reduction). Then, we construct their $\mathbf{L}$ matrix as in eq. (8) by using $\mathbf{Y}_R$ and $\mathbf{Y}_R^{84\%}$ and compute their eigenvalues, respectively.

For the sparsified $\mathbf{L}$, we estimate the upper-bound of eigenvalue errors (ϵ) defined in (9). The error is bounded by $\epsilon = 0.6159$ and it is much smaller for most eigenvalues. The eigenvalue comparison results are shown in Fig. 5(a). Even after the high rate reduction, it captures the eigenvalue property of the original $\mathbf{L}$ to distinguish inter- and intra-area generators.

Additionally, Alg. 2 has the capability to control the error-bound ϵ by varying the edge sampling number, q. We estimate the variation of ϵ with changing q, and the result is plotted in Fig. 6. After applying curve-fitting, we observe that the error-bound decreases roughly proportionally to *inverse-square* with respect to the number of sampled edges, and this matches well with the proof of Alg. 1 in [7].

We also test our sparsification approach on IEEE 30 and 118 Bus systems which have 6 and 34 generators, respectively. Fig. 5(b)-(c) exhibit plots of eigenvalues with varying



(a) IEEE 14 BUS (b) IEEE 30 BUS (c) IEEE 118 BUS

Fig. 7. Eigenvalue comparison results of several electricity grids for stability analysis between an original $A_{\mathbf{sys}}$ and its sparsification by using Alg. 2.

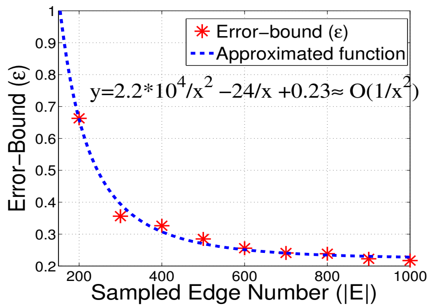


Fig. 6. An error-bound control capability test in 50 machine system: We measure the error-bound with varying a sampling number (q). In Alg.1 [7], an error-bound is controlled by $|E'| = O(n \log n/\epsilon^2)$ and $|E'| \leq q$.

sparsification degree, $50, 70$ and 85%. They also show that eigenvalues on sparsified matrices still preserve the values of complete $\mathbf{L}$ within a small ϵ bound. These observations indicate that our sparsification approach can provide the system mode detection of power grids with a small error even under the use of much simpler structures.

3) Stability Detection Tests from A_{sys}: For stability analysis, we measure eigenvalues from $A_{\mathbf{sys}}$ with damping terms for the IEEE $14, 30$ and 118 Bus systems. Machine damping terms are set to 2 (p.u.) which is the value for the IEEE 14 Bus model in PSAT [18]. To estimate sparsification effects, we use an admittance matrix, 50% sparsified for the IEEE 14 BUS and 70% sparsified for the IEEE 30 and 118 Bus. In Fig. 7, we plot complex eigenvalues for each system. The measured error-bounds (ϵ) for the real-part of eigenvalues are $0.0852, 0.4788, 0.1483$ for the IEEE $14, 30$ and 118 Bus systems, respectively. Even under a high degree of simplification, the structural sparsification of power grids captures real eigenvalues with small errors. Also, we see that the error bound tends to be tighter under the same sparsification rate as the size of power grids increases. Thus, the proposed sparsification approach appears to have increasing advantages in large-scale electricity grids.

V. Conclusion

For better deployment of smart-grid technologies, a fast and scalable electricity grid analysis method remains a critical challenge. The eigenvalues of the system matrix provide an important criterion to determine generator dynamics in small-signal stability analysis. However, their computation requires one to solve a dense system matrix in electricity grids with algebraic buses.

By employing a spectral sparsifier, we provided a network reduction method whose spectral properties are preserved within an ϵ error bound. Our method is tailored to measure not only system modes, but also for stability decisions in power grids, which can fully describe the generator dynamics. Moreover, the sparsification degree (i.e., computational complexity) can be controlled by the tuning of error-bounds. We tested our method on several electricity grid test-beds. The results showed that the spectral property of grids is preserved with only a small error bound even under high-rate sparsifications while it becomes tighter with an increase in grid size.

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