Relaxed Matching for Stabilization of Relative Equilibria of Mechanical Systems

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***Abstract*— Relaxed matching techniques for stabilization of relative equilibria are developed for continuous-time mechanical systems with symmetry that fail to satisfy classical matching conditions. New terms in the controlled shape equations necessary for carrying out relaxed matching are introduced. The theory is illustrated with the problem of stabilization of the steady-state motions of an inverted pendulum on a rotor arm.**

I. Introduction

The method of controlled Lagrangians for stabilization of relative equilibria (steady state motions) originated in Bloch, Leonard, and Marsden [6] and was then developed in Auckly [2], Bloch, Leonard, and Marsden [7], [9], [10], Bloch, Chang, Leonard, and Marsden [11], and Hamberg [14], [15]. A similar approach for Hamiltonian controlled systems was introduced and further studied in the work of Blankenstein, Ortega, van der Schaft, Maschke, Spong, and their collaborators (see, e.g., [18], [19], and related references). The two methods were shown to be equivalent in [12]. For related results on applications of generalized canonical transformations to stabilization of Hamiltonian systems see [13] and references therein. A nonholonomic version of the method of controlled Lagrangians was developed in [20], [21], and [3]. The method was extended to the discrete setting in [4] and [5].

In the controlled Lagrangian approach, one considers a mechanical system with an uncontrolled (free) Lagrangian equal to kinetic energy minus potential energy. This Lagrangian is invariant with respect to the action of a Lie group G on the configuration space. In order to stabilize a relative equilibrium, the kinetic energy is modified to produce a *controlled Lagrangian* which describes the dynamics of the controlled closed-loop system. The equations corresponding to this controlled Lagrangian are the closed-loop equations. The new terms appearing in those equations corresponding to the directly controlled variables are interpreted as control inputs. The modifications to the Lagrangian are chosen so that no new terms appear in the equations corresponding to the variables that are not directly controlled. We refer to this process as *kinetic shaping*. Once the form of the control law is derived using the controlled Lagrangian, the stability of a relative equilibrium of the closed-loop system can be determined by energy methods, using any available freedom in the choice of the parameters of the controlled Lagrangian. To obtain asymptotic stabilization, dissipation-emulating terms are added to the control input.

In order to proceed with kinetic shaping, one needs to verify the *matching conditions* that ensure that the original controlled mechanical system is identical to the system associated with the controlled Lagrangian. These conditions restrict the choice of the modified kinetic energy. It is not always possible to satisfy the matching conditions (see [2], [7], [9], [10], [11] for details). Thus, it is not always possible to construct a stabilizing controller using the method of controlled Lagrangians for an underactuated system.

In this paper we suggest the following modification of kinetic shaping: The dynamics associated with the controlled Lagrangian is amended by non-conservative forces that act in the shape directions. As in the method of controlled Lagrangian, we require that this dynamics is identical to the original controlled dynamics. We show that our approach is less restrictive than the original matching techniques. We carry out the relaxed matching procedure explicitly for systems with one shape and one group degree of freedom in order to avoid technical issues and to concentrate on the phenomena that emerge in the new setting.

The theoretical analysis is validated by simulating the inverted pendulum on a rotor arm. When dissipation is added, the inverted pendulum configuration is asymptotically stabilized, as predicted.

In a forthcoming publication we intend to extend our formalism to systems with nonabelian symmetries and to full state space stabilization problems.

The paper is organized as follows: In Section II we review the method of controlled Lagrangians for stabilization of relative equilibria of mechanical systems. The main results of the paper are exposed in Section III. Stabilization of the pendulum on a rotor arm and simulations are presented in Section IV.

II. Matching and Controlled Lagrangians

In this section we briefly review some of the results of Bloch, Leonard, and Marsden [10]. We limit this review to the instance of one shape and one group degree of freedom. We further assume that the configuration space Q is the direct product of a one-dimensional shape space S and a one-dimensional Lie group G.

The configuration variables are written as (ϕ, s), with $\phi \in S$ and $s \in G$. The velocity phase space, TQ, has coordinates $(\phi, s, \dot\phi, \dot s)$. We assume that the Lagrangian is G-invariant and reads

$$L(\phi,\dot\phi,\dot s) = \tfrac{1}{2}\big[\alpha\dot\phi^2 + 2\beta(\phi)\dot\phi\dot s + \gamma(\phi)\dot s^2\big] - U(\phi),^{1} \quad (1)$$

and the corresponding dynamics is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} - \frac{\partial L}{\partial\phi} = 0, \quad (2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot s} = u, \quad (3)$$

where u is the control inputs.

Recall that a *relative equilibrium* of a system with symmetry is a solution whose reduced trajectory is an equilibrium of the reduced equations.

We assume that the relative equilibria $\phi = \phi_e$, $\dot s = \text{const}$ of uncontrolled system (2) and (3) are unstable. In the rest of the paper we assume that $\phi_e = 0$, which can always be accomplished by an appropriate choice of local coordinates for each relative equilibrium. In order to stabilize the relative equilibria $\phi = 0$, $\dot s = \text{const}$, Bloch, Leonard, and Marsden [10] define the controlled Lagrangian by

$$L_{\tau,\sigma}(\phi,\dot\phi,\dot s) = L(\phi,\dot\phi,\dot s + \tau(\phi)\dot\phi) + \tfrac{1}{2}\sigma(\phi)(\tau(\phi)\dot\phi)^2. \quad (4)$$

The velocity shift $\dot s \to \dot s + \tau(\phi)\dot\phi$ corresponds to a new choice of the horizontal space, while the last term in (4) changes the metric along the vertical (*i.e.,* tangent to the group orbit) direction (see [10] for details).

The major result of [10] is the following theorem (recall that here we only state the results for systems with one shape and one group degree of freedom).

Theorem 1: The controlled Euler–Lagrange equations (2) *and* (3) *coincide with the Euler–Lagrange equations for the controlled Lagrangian* (4) *if*

$$u = -\frac{d}{dt}\big(\gamma(\phi)\tau(\phi)\dot\phi\big)$$

and the following ***matching conditions*** *hold:*

$$\sigma(\phi)\tau(\phi) = -\beta(\phi), \quad (5)$$

$$(\sigma'(\phi) + \gamma'(\phi))/\sigma(\phi) = 2\gamma'(\phi)/\gamma(\phi), \quad (6)$$

$$\gamma'(\phi)\tau(\phi) = 0. \quad (7)$$

The quantities $\tau(\phi)$ and $\sigma(\phi)$ are selected in such a way that the relative equilibria of interest become orbitally stable, that is, one observes stability relative to the variables ϕ, $\dot\phi$, and $\dot s$, but not relative to s.

Matching conditions (5)–(7) imply that $\gamma(\phi) = \text{const}$ generically. In order to use the matching techniques in the case of non-constant $\gamma(\phi)$, Bloch, Leonard, and Marsden introduced a more general controlled Lagrangian

$$L_{\tau,\sigma,\rho}(\phi,\dot\phi,\dot s) = L(\phi,\dot\phi,\dot s + \tau(\phi)\dot\phi) + \tfrac{1}{2}\sigma(\phi)(\tau(\phi)\dot\phi)^2 + \tfrac{1}{2}(\rho(\phi) - \gamma(\phi))(\dot s + \beta(\phi)\dot\phi/\gamma(\phi) + \tau(\phi)\dot\phi)^2.$$

[1]The coefficient α in (1) is assumed to be independent of ϕ. This is done in order to simplify the exposition.

The presence of an extra term leads to a less restrictive set of matching conditions (see [8], [9], [10], [11] for details).

In this paper we introduce an alternative approach to the problem of stabilization of relative equilibria of (2) and (3) in the case of non-constant metric coefficient $\gamma(\phi)$. While we develop the theory for systems with one shape and one group degree of freedom, it will be clear from the exposition that the method is general and that it can be applied to a class of systems that is wider than systems stabilizable by the original method of controlled Lagrangians.

III. Relaxed Matching

A. Matching with Shape Forcing

The key idea of relaxing matching techniques is to introduce a non-conservative force in the shape equation associated with the controlled Lagrangian (4). That is, the dynamics associated with (4) is

$$\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial\dot\phi} - \frac{\partial L_{\tau,\sigma}}{\partial\phi} = w, \quad (8)$$

$$\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial\dot s} = u_{\text{diss}}, \quad (9)$$

where $u_{\text{diss}} = c(\phi)\dot\phi$ is the *dissipation-emulating* term that is necessary for asymptotic stabilization. Let $C(\phi)$ be an antiderivative of $c(\phi)$ that vanishes at $\phi = \phi_e$. In the rest of the paper the dependence of the quantities $C(\phi)$, $c(\phi)$, $\beta(\phi)$, $\gamma(\phi)$, $\sigma(\phi)$, and $\tau(\phi)$ on ϕ is not always written explicitly.

Theorem 2: The controlled Euler–Lagrange equations (2) *and* (3) *coincide with equations* (8) *and* (9) *associated with the controlled Lagrangian* (4) *if*

$$u = -\frac{d}{dt}\big(\gamma\tau\dot\phi\big) + c\dot\phi, \quad (10)$$

$$w = \big(\beta\tau + \sigma\tau^2\big)\ddot\phi - \gamma'\tau(p + C)\dot\phi/\gamma + \tau c\dot\phi + \big(\beta\gamma'\tau/\gamma + \beta\tau' + \tfrac{1}{2}\gamma'\tau^2 + \tfrac{1}{2}(\sigma\tau^2)'\big)\dot\phi^2. \quad (11)$$

Proof: We start by asking that equation (3) is identical to equation (9). This is accomplished by setting

$$u = \frac{d}{dt}\frac{\partial L}{\partial\dot s} - \frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial\dot s} + c\dot\phi.$$

Straightforward calculation shows that this requirement is equivalent to condition (10), which defines the control input u for system (2) and (3).

Equation (9) is equivalent to the conservation law

$$\frac{\partial L_{\tau,\sigma}}{\partial\dot s} - C(\phi) = p, \quad (12)$$

where

$$\frac{\partial L_{\tau,\sigma}}{\partial\dot s} = \beta\dot\phi + \gamma\dot s + \gamma\tau\dot\phi$$

is the *controlled momentum*, and the constant p labels the levels of the conservation law (12).

To finish the proof, we require that equations (2) and (8), restricted to *controlled momentum levels*, are the same. This defines the term w by the formula

$$w = \left[\left(\frac{d}{dt}\frac{\partial L_{\tau,\sigma}}{\partial\dot\phi} - \frac{\partial L_{\tau,\sigma}}{\partial\phi}\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} - \frac{\partial L}{\partial\phi}\right)\right]_{\partial_{\dot s}L_{\tau,\sigma} - C = p},$$

which is equivalent to (11), as a direct calculation shows. ■

Remark. Condition (11) replaces the matching conditions (5)–(7).

B. Reduced Dynamics and Stability Conditions

Recall that equations (8) and (9) have a conservation law (12). We now compute the reduced dynamics, *i.e.,* the dynamics on the level sets of conservation law (12). We will see that the reduced dynamics has the structure of the forced Euler–Lagrange equations, which is important for stability analysis.

Define the reduced Lagrangian by the formula

$$l_p(\phi, \dot{\phi}) = \frac{1}{2}\Big(\alpha - \frac{\beta^2}{\gamma} - \beta\tau\Big)\dot{\phi}^2 - U_p, \tag{13}$$

where

$$\begin{aligned} U_p(\phi) &= U(\phi) - \int_0^{\phi} \frac{\gamma'(C(x)+p)^2}{2\gamma^2(x)}\, dx \\ &= U(\phi) + \frac{p^2}{2\gamma(\phi)} - \int_0^{\phi} \frac{\gamma' C(x)(C(x)+2p)}{2\gamma^2(x)}\, dx \end{aligned}$$

is the *amended potential*. Let f be a (non-conservative) force defined by

$$\begin{aligned} f = &-\frac{\gamma'}{\gamma}(p+C)\tau\dot{\phi} \\ &+ \Big(\frac{1}{2}(\beta\tau' - \beta'\tau) + \frac{1}{2}\gamma'\tau^2 + \frac{\beta\gamma'\tau}{\gamma}\Big)\dot{\phi}^2. \end{aligned}$$

The following statement is obtained by a straightforward caclulation.

Theorem 3: Dynamics (8) *and* (9) *reduced to level sets of conservation law* (12) *is given by the forced Euler–Lagrange equations for the reduced Lagrangian* (13)*,*

$$\frac{d}{dt}\frac{\partial l_p}{\partial \dot{\phi}} - \frac{\partial l_p}{\partial \phi} = -\frac{\beta c}{\gamma}\dot{\phi} + f. \tag{14}$$

Remark. Neither the reduced Lagrangian, nor control input (10) depends on the term $\frac{1}{2}\sigma(\tau\dot{\phi})^2$ in the controlled Lagrangian (4). Thus, without loss of generality we can set $\sigma = 0$. Of course, in the case when the standard matching techniques are applicable, one may be motivated to select σ that satisfies the matching conditions (5)–(7) as this value of σ eliminates force f in (14).

Recall that $\phi = 0$, $\dot{s} = \text{const}$ are relative equilibria of (2) and (3). That is, $\phi = 0$ is an equilibrium of the reduced shape equation (14). If this equilibrium is stable, the corresponding relative equilibria of (2) and (3) are *orbitally stable*.

The energy associated with Lagrangian (13) is

$$E_p(\phi, \dot{\phi}) = \frac{1}{2}\Big(\alpha - \frac{\beta^2}{\gamma} - \beta\tau\Big)\dot{\phi}^2 + U_p. \tag{15}$$

We now show that (15) can be used as a Lyapunov function for stability analysis of relative equilibria $\phi = 0$, $\dot{s} = \text{const}$.

Recall that the relative equilibria of interest of the uncontrolled system are unstable, which implies

$$\frac{d^2}{d\phi^2}\Big(U + \frac{p^2}{2\gamma}\Big)(0) \le 0.$$

We assume here that $U_p''(0)$ is negative.[2] We then select $\tau(\phi)$ such that

$$\Big(\alpha - \frac{\beta^2}{\gamma} - \beta\tau\Big) < 0, \tag{16}$$

which makes reduced energy (15) *negative-definite* in a neighborhood of the equilibrium of interest. The flow derivative of E_p is

$$\dot{E}_p = -\frac{\beta c}{\gamma}\dot{\phi}^2 + f\dot{\phi}.$$

Thus, selecting $c(\phi)$ such that

$$\frac{\beta c}{\gamma} + \frac{\gamma'(p+C)\tau}{\gamma} < 0 \tag{17}$$

makes $\dot{E}_p$ non-negative in a neighborhood of the equilibrium of interest. LaSalle's invariance principle can then be used to establish asymptotic stability of the equilibrium $\phi = 0$ of (14) and the size of the basin of attraction. Summarizing, we have the following result.

Theorem 4: The equilibrium $\phi = 0$ of the reduced shape equation (14) *is asymptotically stable if $U_p''(0)$ is negative and conditions* (16) *and* (17) *hold.*

IV. Stabilization of the Pendulum on a Rotor Arm

Consider a planar pendulum attached to a horizontal rotor arm as shown in Figure 1. This mechanical system is studied in Åström and Furuta [1] and Bloch, Leonard, and Marsden [8]. The latter paper shows that matching conditions (5)–(7) cannot be satisfied for the pendulum on a rotor arm.

The plane of the pendulum is orthogonal to the arm. The rotor arm is subject to a control torque u that we intend to use for stabilizing the upward vertical relative equilibrium of the pendulum. As shown in the figure, the length of the pendulum is l, the pendulum bob mass is m, the length of the rotor arm is R, the mass attached to the rotor arm is M, and the tilt of the pendulum measured from the upward vertical is ϕ. The orientation of the rotor arm is given by the angle s. The configuration space for this system is the two-dimensional torus parametrized by the angles ϕ and s.

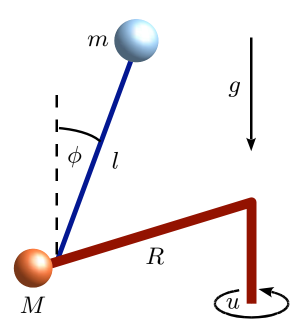

Fig. 1. The pendulum on a rotor arm.

[2]This is true for the pendulum on a rotor arm and, more generally, for systems with $\gamma'(\phi)$ vanishing at the relative equilibria.

The Lagrangian for this system is given by formula (1) with the kinetic energy metric coefficients α, $\beta(\phi)$, and $\gamma(\phi)$ defined by the formulae

$$\begin{aligned}\alpha &= ml^2,\\ \beta(\phi) &= mlR\cos\phi,\\ \gamma(\phi) &= ml^2\sin^2\phi + (m+M)R^2,\end{aligned}$$

and the potential energy given by the formula

$$U(\phi) = mgl\cos\phi.$$

This system is invariant with respect to rotations about the axis of the rotor arm, *i.e.,* s is a cyclic variable. The relative equilibria of the unforced ($u = 0$) system are

$$\phi = \phi_e, \quad \dot\phi = 0, \quad s = \omega t + s_0, \quad \dot s = \omega,$$

where $\omega = \mathrm{const}$ and where ϕ_e are roots of the equation

$$\sin\phi\left(\omega^2\cos\phi + \frac{g}{l}\right) = 0, \tag{18}$$

see [8] for details. Equation (18) has two solutions $\phi_e = 0, \pi$ if $\omega^2 < g/l$. When $\omega^2 > g/l$, two additional solutions $\phi_e = \pm\arccos(-g/(\omega^2 l))$ appear; the corresponding relative equilibria are stable. The upright vertical relative equilibrium $\phi_e = 0$ is always unstable. The relative equilibrium $\phi_e = \pi$ is stable if $\omega^2 < g/l$. It becomes unstable when $\omega^2 > g/l$.

We now stabilize the upward vertical relative equilibrium

$$\phi_e = 0, \quad \dot s = \omega \tag{19}$$

of the pendulum on a rotor arm using techniques developed in Section III. Note that condition (16) fails if $\phi = \pm\pi/2$, and thus the range of ϕ, for any choice of $\tau(\phi)$, cannot exceed the interval $(-\pi/2, \pi/2)$.

Below we assume that $c(\phi)$ is a negative constant, c. The amended potential and its derivative for the pendulum on the rotor arm are

$$\begin{aligned}U_p &= mgl\cos\phi - mgl\\ &\quad - \int_0^\phi \frac{ml^2\sin x\cos x\,(p+cx)^2}{(ml^2\sin^2 x + (m+M)R^2)^2}\,dx, \qquad (20)\\ U_p' &= -\sin\phi\Big[mgl + \frac{ml^2\cos\phi\,(p+c\phi)^2}{(ml^2\sin^2\phi + (m+M)R^2)^2}\Big]. \qquad (21)\end{aligned}$$

Formulae (20) and (21) imply that the amended potential $U_p(\phi)$ is negative-definite throughout the interval $(-\pi/2, \pi/2)$ and has a single maximum at $\phi = 0$.

Define $\tau(\phi)$ by the formula

$$\tau(\phi) = \kappa\frac{\beta(\phi)}{\gamma(\phi)}. \tag{22}$$

Condition (16) becomes

$$\frac{(M - \kappa m)R^2 + \big(mR^2(1+\kappa) + ml^2\big)\sin^2\phi}{(m+M)R^2 + ml^2\sin^2\phi} < 0.$$

The latter holds if

$$|\phi| < \arcsin\sqrt{\frac{\kappa - M/m}{\kappa + 1 + l^2/R^2}}. \tag{23}$$

Since

$$\sqrt{\frac{\kappa - M/m}{\kappa + 1 + l^2/R^2}} < 1 \quad \text{and} \quad \lim_{\kappa\to\infty}\sqrt{\frac{\kappa - M/m}{\kappa + 1 + l^2/R^2}} = 1,$$

interval (23) approaches $(-\pi/2, \pi/2)$ as $\kappa \to \infty$. Thus, the proposed controller results in a larger stabilization region for ϕ than

$$|\phi| < \arcsin\sqrt{\frac{R^2}{R^2 + l^2}}, \tag{24}$$

which is the maximal possible range one can obtain by using the original matching procedure (see [8] for details).

Stability conditions (16) and (17) for this controller are satisfied if ϕ belongs to interval (23) and $|p|$ is not too large. Therefore, the upward vertical relative equilibrium (19) becomes stable. The simulation results for the pendulum on a rotor arm with $m = 1\,\mathrm{kg}$, $M = 2\,\mathrm{kg}$, $l = 1\,\mathrm{m}$, and $R = 2\,\mathrm{m}$ are given in Figure 2. The initial conditions and numerical values of the gain parameters in this simulation are $\phi(0) = \pi/4\,\mathrm{rad}$, $\dot\phi(0) = 0\,\mathrm{rad/s}$, $s(0) = 0\,\mathrm{rad}$, $\dot s(0) = 0\,\mathrm{rad/s}$, $c = -10\,\mathrm{N\cdot m\cdot s}$, and $\kappa = 2/3$.

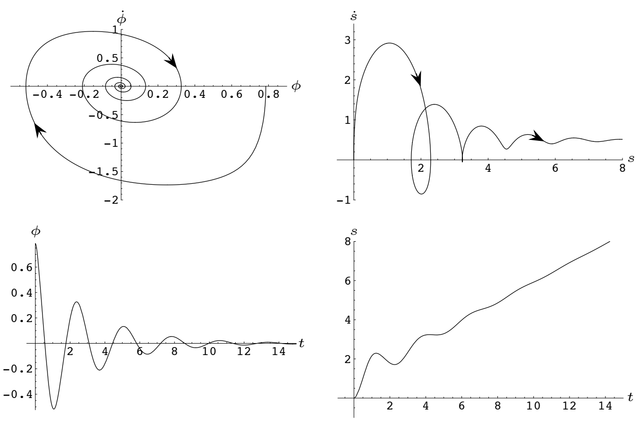

Fig. 2. Asymptotic stabilization of a pendulum from $\phi(0) = \pi/4$; $\tau(\phi)$ is given by formula (22).

The torque produced by this controller is

$$u = -\frac{d}{dt}\big(\kappa\beta(\phi)\dot\phi\big) + c\dot\phi = -\frac{d}{dt}\big(\kappa mlR\cos\phi\,\dot\phi\big) + c\dot\phi. \tag{25}$$

Formulae (23) and (25) suggest that one needs a large value of κ if the goal is to stabilize the pendulum from a near-horizontal initial position.

Since a large value of the gain parameter κ is not always desirable, we suggest below a different selection for $\tau(\phi)$.

Define $\tau(\phi)$ by the formula

$$\tau(\phi) = \varkappa\frac{ml^2}{\beta(\phi)}. \tag{26}$$

In this case stability condition (16) becomes

$$ml^2 - ml^2\varkappa - \frac{m^2l^2R^2\cos^2\phi}{(m+M)R^2 + ml^2\sin^2\phi} < 0.$$

This stability condition is satisfied for the entire interval $|\phi| < \pi/2$ if

$$\varkappa > 1.$$

Stability condition (17) is satisfied if $|\phi| < \pi/2$ and $|p|$ is not too large. The torque produced by this controller is

$$u = -\frac{d}{dt}\left[\frac{\varkappa\gamma(\phi)ml^2\dot{\phi}}{\beta(\phi)}\right] + c\dot{\phi}$$
$$= -\frac{d}{dt}\left[\frac{\varkappa\left[ml^2\sin^2\phi + (M+m)R^2\right]\dot{\phi}}{R\cos\phi}\right] + c\dot{\phi}. \quad (27)$$

Since the denominator in (27) vanishes as $\phi \to \pm\pi/2$, formula (27) suggests that this controller may be capable of stabilizing the upward relative equilibrium from near horizontal initial tilt of the pendulum even if $\varkappa$ is not too large. This is confirmed by numerical simulation as discussed below.

The gain parameters for simulation results in Figures 3 and 4 are $c = -50\ \mathrm{N\cdot m\cdot s}$ and $\varkappa = 8/5$.

Figure 3 demonstrates stabilization of the pendulum by control torque (27) from the state $\phi(0) = \pi/4\ \mathrm{rad}$, $\dot{\phi}(0) = 0\ \mathrm{rad/s}$, $s(0) = 0\ \mathrm{rad}$, $\dot{s}(0) = 0\ \mathrm{rad/s}$. The convergence rate appears to be faster than in Figure 2, possibly because of the value of the coefficient c.

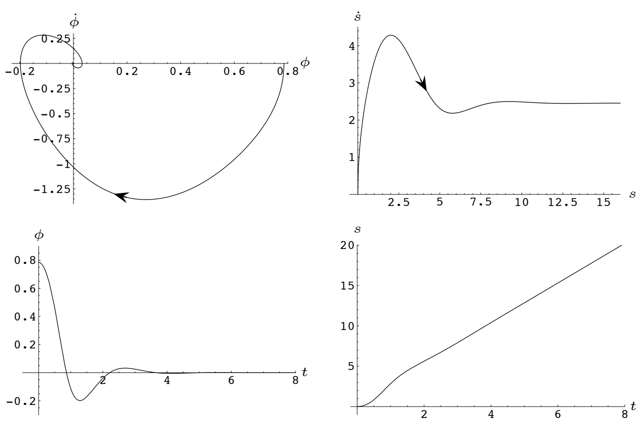

Fig. 3. Asymptotic stabilization of a pendulum from $\phi(0) = \pi/4$; $\tau(\phi)$ is given by formula (26).

Figure 4 demonstrates stabilization of the pendulum by control torque (27) from a nearly horizontal position $\phi(0) = \pi/2 - 0.02\ \mathrm{rad}$. The remaining initial conditions are $\dot{\phi}(0) = 0\ \mathrm{rad/s}$, $s(0) = 0\ \mathrm{rad}$, $\dot{s}(0) = 0\ \mathrm{rad/s}$.

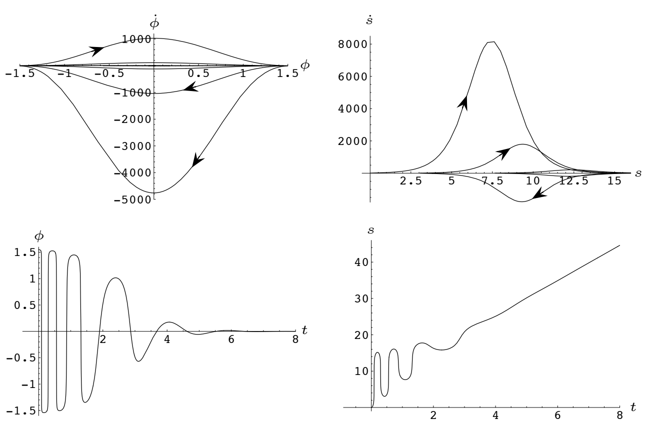

Fig. 4. Asymptotic stabilization of a pendulum from $\phi(0) = \pi/2 - 0.02$; $\tau(\phi)$ is given by formula (26).

Note that the controller in [8] cannot stabilize the pendulum from the initial tilt $\phi(0) = \pi/2 - 0.02\ \mathrm{rad}$ as this tilt fails to satisfy condition (24). The latter reads $|\phi| < \arcsin(2/\sqrt{5}) \approx 1.10715$ for the system parameters used in the simulations.

V. Conclusions

This paper has introduced relaxed matching techniques for mechanical systems with symmetry and has shown that these lead to an effective stabilizing controller design. Suggested formalism makes use of the intrinsic structure of mechanical systems and is less restrictive than the original matching procedure. Systems with non-commutative symmetry and combined kinetic and potential shaping for full state space stabilization will be treated in a forthcoming publication.

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