1. (15 pts each) page 192–205: 5, 10, 17, 41, 54

- Problem 5.

**Solution:** If \( n \) is even then the nodes are \( x_j = -1 + (j - 1)h \) for \( 0 \leq j \leq n \). This means that \( x_{n/2} = 0 \) is a node in the rule. Hence \( T(n/2) \) is the same as integrating \(-x\) from \(-1\) to \(0\) and \(x\) from \(0\) to \(1\), which the trapezoid rule does exactly.

When \( n \) is odd, \( 0 \) is not a node. The error is

\[
\int_{-h/2}^{h/2} |x| \, dx - h \frac{h}{2} = \frac{h^2}{4}.
\]

The error is second order, but the constant in the O-term is \( 1/4 \) and not \( h^2 \| f'' \|_\infty /6 \) (which is not defined because \( f \) is not differentiable at \( 0 \)).

- Problem 10.

The one is easy. All I have do is evaluate the terms in the errors.

**Solution:** (a) The error in the trapezoid rule for this problem is

\[
\frac{-h^2 f''(\xi)}{12}
\]

for some \( \xi \in (0, 1) \). We will use the standard estimate

\[
\|E\| \leq \frac{h^2 \| f'' \|_\infty}{12}.
\]

Since

\[
f'(x) = -2xe^{-x^2} \text{ and } f''(x) = -2e^{-x^2} + 4xe^{-x^2} = 2e^{-x^2}(-1 + 2x^2)
\]

we can compute the maximum of \( |f''| \) with calculus. We have

\[
f'''(x) = -4xe^{-x^2}(-1 + 2x^2) + 8xe^{-x^2} = e^{-x^2}(12x - 8x^3)
\]

has its only root in \([0, 1]\) at \( x = 0 \). \( f''(0) = -2 \), but we have to check the other endpoint. \( f''(1) = 4/e < 2 \), so \( \|f''\|_\infty = 2 \). So we now need to find the smallest \( n \) such that

\[
\frac{(1/n)^22}{12} < 10^{-6}/2
\]

and then we will need \( n \) intervals. So we need \( N > \sqrt{1/310^3} \approx 577.3 \). The answer is 578.
Solution: (b) It’s the same process. The Simpson’s rule error is
\[
-\frac{h^4 f^{iv}(\xi)}{180}.
\]
I will need to maximize
\[
f^{iv}(x) = -2xe^{-x^2}(12x - 8x^3) + e^{-x^2}(12 - 24x^2) \\
= e^{-x^2}(12 - 48x^2 + 16x^4). = 4e^{-x^2}(3 - 12x^2 + 4x^4).
\]
It’s clear that \( f^{iv} \) is a decreasing function, so the maximum of \(|f^{iv}|\) is at one of the endpoints. \( f^{iv}(0) = 12 \) and \( f^{iv}(1) = -20/e \), so \( \|f^{iv}\|_{\infty} = 12 \). So, I need to find \( n \) so that
\[
\frac{n^{-4}12}{180} < 10^{-6}/2
\]
which means \( n > (210^6/15)^{1/4} \approx 19.1 \), so the answer is 20.

Problem 17.

Solution: (a) The spline has degree three on each side and is continuous. The derivative is not continuous, so \( s \in S_{S_{3}}^0 \).

Solution: (b) The function is a polynomial of degree 3 on both halves of the interval, and the trapezoid rule will not see the discontinuity in the derivative because (as in a previous problem) the discontinuity is at a node. So the error estimate is
\[
\frac{h^22\|f''\|_{\infty}}{12} = h^2.
\]

Solution: (c) It’s the same story. We will use the error estimate
\[
\frac{h^42\|f^{iv}\|_{\infty}}{180} = 0
\]
because Simpson’s rule integrates cubics exactly and the singularity at \( x = 0 \) is not in the interior of any interval (except for \( n = 1 \)).

Solution: (d) Two point Gauss integrates cubic polynomials exactly and \( s \) is therefore integrated exactly on each interval. The error is zero.
Problem 41.

**Solution:** (a) I will use the method of undetermined coefficients. This one is a direct calculation. Do the integrals to see that

\[ a + b = 1, \quad b = 1/2, \quad \text{and} \quad b - 2c = 1/3. \]

So, \( a = b = 1/2 \) and \( c = 1/12 \).

**Solution:** (b) All I need to do is move the nodes and scale the weights. The nodes go to \( x \) and \( x + h \) and the weights are scaled by \( h \), the length of the new interval. So the rule is

\[
\frac{h}{2} (f(x) + f(x + h)) - \frac{h^2}{12} (f'(x + h) - f'(x))
\]

Note the extra power of \( h \) in the \( f' \) term.

**Solution:** (c) I will use a constant stepsize with \( x_i = a + (i-1)h \) and \( h = (b-a)/n \). The rule is

\[
\int_a^b f(x) \, dx = \sum_{i=1}^{n} \frac{h}{2} (f(x_i) + f(x_{i+1})) - \sum_{i=1}^{n} \frac{h^2}{12} (f'(x_i) + f'(x_{i+1}))
\]

The first sum is the composite trapezoid rule and the second (see the 12) cancels the leading order error term (can you see how?).

Problem 54.

**Solution:** (a) This is yet another undetermined coefficients problem. Do the integrals with \( f(x) = 1 \) and \( f(x) = x \) and you get

\[
\frac{1}{1 + \alpha} = \alpha_1 + \frac{\alpha_2}{2}
\]

and

\[
\frac{1}{2 + \alpha} = \frac{\alpha_1}{2} + \frac{\alpha_2}{3}.
\]

You can solve this with brute force elimination. Subtract twice the second equation from the first and get

\[
\frac{1}{1 + \alpha} - \frac{2}{2 + \alpha} = \frac{\alpha_2}{2} - \frac{2\alpha_2}{3} = -\alpha_2/6.
\]

having solved for \( \alpha_2 \), plug in to get \( \alpha_1 \).
2. Programming Assignment

(25 pts) Do a grid refinement study to examine the accuracy of some non-adaptive composite integration rules. For both of the composite midpoint and composite 4-point Gauss rules do the following.

For \( h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{256} \) integrate each member of the list of functions below on \([0, 1]\). Examine the errors \( E_h(f) \) and the ratios of errors \( E_h(f)/E_{2h}(f) \). Would you expect the ratios to converge? To what value? If any of these functions fails to meet your expectation, explain why.

(a) \( f(x) = e^x \)
(b) \( f(x) = x^2 \ln(x) \)
(c) \( f(x) = \sin(5\pi x) \)

Present your results as a table (in short e format) where \( h, E_h(f), \) and \( E_h(f)/E_{2h}(f) \) are the columns. I’m expecting 7 columns.

Solution: The precision of four point Gauss is 7, so I’d expect an order of 8. Hence halving the mesh size should reduce the error by a factor of 256. The midpoint rule is 2nd order, so I’d expect to see reductions by a factor of four. I see exactly what I expect in the midpoint rule table, but not in the Gauss table.

I wrote a code prog2.m to loop through the problems and make the table. The function \( e^x \) is very easy to integrate, so I used a very large value of \( h \) at the beginning to 1. I see something pretty close to 256 for the first two values of \( h \), then the accuracy is at the level of machine roundoff, so there’s no more reduction. A similar thing happens for \( \sin(5\pi x) \). For the second problem, I see a clear reduction of a factor of 8, telling me the rule is only 3rd order accurate. This makes sense because \( x^2 \ln(x) \) has only two continuous derivatives. The 3rd derivative is integrable, however, so when you measure the accuracy of an integral, you’ll see 3rd order accuracy.

My codes are midptc.m and gauss4.m. The functions and driver program are fun1.m, fun2.m, fun3.m and prog2.m.
Table 1: Midpoint Rule

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<tr>
<th></th>
<th>( e^x )</th>
<th>( x^2 \ln(x) )</th>
<th>( \sin(5\pi x) )</th>
</tr>
</thead>
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<tr>
<td>( h )</td>
<td>( \varepsilon_h(f) )</td>
<td>( \varepsilon_{2h}(f)/\varepsilon_h(f) )</td>
<td>( \varepsilon_h(f) )</td>
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<td>9.33e–10</td>
<td>0.00e+00</td>
<td>4.74e–05</td>
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Table 2: Four-point Gauss Rule

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<th>( x^2 \ln(x) )</th>
<th>( \sin(5\pi x) )</th>
</tr>
</thead>
<tbody>
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<td>( h )</td>
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<td>( E_{2h}(f)/E_h(f) )</td>
<td>( E_h(f) )</td>
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