MA780, sec 001, test 1, Feb 24, 2010

Solutions

1. (25 points) Let $f \in C^4[0,1]$ and let $p$ be the cubic polynomial which interpolates $f(0)$, $f'(0)$, $f''(0)$, and $f(1)$. State and prove the error formula for $E = f - p$.

**Solution:** This is right from the book and class notes. The answer is

$$E = \frac{f^{(4)}(\xi)}{4!}x^3(x-1).$$

The proof uses the same auxiliary function argument that we used in class several times. Let $x \in (0,1)$ and let

$$F(z) = f(z) - p(z) - \frac{f(x) - p(x)}{x^3(x-1)}z^3(z-1).$$

By construction, $F(0) = F(x) = F(1) = 0$. Rolle’s theorem says there are $\xi_1 \in (0, x)$ and $\xi_2 \in (x, 1)$ such that so that $F'(\xi_j) = 0$ for $j = 1, 2$. Since $F'(0) = 0$, there are $\xi_3 \in (0, \xi_1)$ and $\xi_4 \in (\xi_1, \xi_2)$ with $F''(\xi_j) = 0$ for $j = 3, 4$. Since $F''(0) = 0$ we get $\xi_5 \in (0, \xi_3)$ and $\xi_6 \in (\xi_3, \xi_4)$ with $F'''(\xi_j) = 0$ for $j = 5, 6$. So, at last we have $\xi \in (\xi_5, \xi_6)$ so that

$$F^{(4)}(\xi) = f^{(4)}(\xi) - \frac{f(x) - p(x)}{x^3(x-1)}4! = 0,$$

and that’s it.
2. (20 points)
   (a) Let $N > 0$. Show that the functions $\{x^i\}^N_{i=0}$ are linearly independent.

   **Solution:** Linear dependence would imply that there are $\{a_i\}^N_{i=0}$ for which
   
   $$p(x) = \sum_{i=0}^{N} a_i x^i \equiv 0$$
   
   which means $p$ would be an $N$th degree polynomial with infinitely many roots. This contradicts the fundamental theorem of algebra.

   (b) Compute the quadratic polynomial that interpolates the data

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$f(x_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

   **Solution:** I’ll use the Newton form and look for
   
   $$p(x) = a(x-1)x + bx.$$ 

   I get $p(0) = 0$ for free. If $b = 1$, then $p(1) = 1$. Finally $p(2) = 2a + 2b = 8$ implies that $a = (8 - 2)/2 = 3$. Bottom line: $p(x) = 3(x-1)x + x$. 
3. (25 points) Use interpolation theory to show that the composite trapezoid rule is second order accurate and derive the error formula. I know we did this in class, and I want you to do it completely anyhow.

**Solution:** Let \( h = (b - a)/N \) and let \( x_i = a + (i - 1)h \) be the left endpoint of one of the subintervals. I’ll apply interpolation theory on \([x_i, x_{i+1}]\) to get

\[
|\int_{x_i}^{x_{i+1}} f(x) \, dx - h \frac{f(x_i) + f(x_{i+1})}{2}| \leq \frac{\|f''\|_{\infty}}{2} \int_{x_i}^{x_{i+1}} |(x - x_i)(x - x_i - h)| \, dx
\]

\[
\leq h^3 \|f''\|_{\infty}. 
\]

Hence, summing the integrals over the subintervals, the error in the composite trapezoid rule is

\[
\leq h^2 \frac{(b - a)\|f''\|_{\infty}}{12}. 
\]

4. (30 points)

(a) Define the Chebychev polynomials.

**Solution:** For \( x = \cos(\theta) \in [-1, 1] \),

\[
T_n(x) = \cos(n\theta)
\]

(b) State and prove the recursion formula for Chebychev polynomials.

**Solution:** The recursion is

\[
T_{k+1} = 2xT_k - T_{k-1}
\]

which you could have either remembered or derived as part of this problem. The solution is to use trig identities, which I can never remember. So I will use \( e^{i\theta} = \cos(\theta) + i \sin(\theta) \) to figure it out (here \( i = \sqrt{-1} \)).

\[
e^{i(\alpha + \beta)} = \cos(\alpha + \beta) + i \sin(\alpha + \beta)
\]

\[
e^{i\alpha}e^{i\beta} = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) + i(\sin(\alpha) \cos(\beta) + \sin(\alpha) \cos(\beta))
\]

Take real parts to get

\[
\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta).
\]

Now apply the equation above twice, once with \( \alpha = k\theta \) and \( \beta = \theta \), then with \( \alpha = k\theta \) and \( \beta = -\theta \), and add the results to get

\[
\cos((k + 1)\theta) + \cos((k - 1)\theta) = 2 \cos(k\theta) \cos(\theta)
\]

and that’s it.
(c) Let $T_n$ be the $n$th monic Chebychev polynomial. Give a formula for $T_n''(0)$.

**Solution:** This is very close to the homework. Take the definition and differentiate once to get

$$\frac{dT_k(\cos(\theta))}{d\theta} = -\sin(\theta)T_k'(\cos(\theta)) = -k \sin(k\theta).$$

Do it again to get

$$-\cos(\theta)T_k'(\cos(\theta)) + \sin^2(\theta)T_k''(\cos \theta) = -k^2 \cos(k\theta).$$

Plug in $\theta = \pi/2$ to get $x = 0$, $\sin(\theta) = 1$ and

$$T_k''(0) = -k^2 \cos(k\pi/2).$$

If $k$ is odd, you get $T_k'' = 0$. If $k = 2n$ is even, then

$$T_{2n}''(0) = -(2n)^2 \cos(n\pi) = 4n^2(-1)^{n+1}.$$